Bilocal Dynamics in Quantum Field Theory

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Abstract

An essential aspect of noncommutative field theories is their bilocal nature. This feature, and its role in the IR/UV mixing, are discussed using a canonical quantization procedure developed recently.

Locality has been long considered as an essential ingredient of Quantum Field Theories, although attempts to go beyond this powerful constraint ocasionally appeared. Recently, a peculiar form of nonlocality atracted attention, in the context of noncommutative (NC) field theories (FT) [1]. Intuitive, stringy or Weyl-Moyal based arguments appeared to favour a dipolar nature of the degrees of freedom of such theories [2].

We will present here a different approach, based on a canonical quantization procedure developed recently [3]. It clearly demonstrates the intrinsic bilocal nature of noncommutative fields, and renders transparent the nature of the real space-time on which dynamics takes place, and on which measurements could be performed (as opposed to the fictitious Weyl symbols space). This approach allows one to view our space from different perspectives [3, 4], corresponding to the representation of the NC algebra one chooses. Comments on the IR/UV mixing are also presented.

Bilocal objects

The simplest NC field is a $(2+1)$-dimensional scalar $\Phi(t, \hat{x}, \hat{y})$, defined over a commuting time t and a pair of NC coordinates which satisfy

$$[\hat{x}, \hat{y}] = i\theta. \tag{1}$$

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The extension to several NC pairs is straightforward. The action is

$$S = \frac{1}{2}\int \mathrm{d}t\, \mathrm{Tr}_{\mathcal{H}}\left[\dot{\Phi}^2 - (\partial_x\Phi)^2 - (\partial_y\Phi)^2 - m^2\Phi^2 - 2V(\Phi)\right]. \qquad (2)$$

We will exemplify with a quartic potential, $V(\Phi) = \frac{g}{4!}\Phi^4$. The operators $\hat{x}$ and $\hat{y}$ act on a harmonic oscillator Hilbert space $\mathcal{H}$ in the usual way. $\mathcal{H}$ may be given a discrete basis $\{|n>\}$ formed by eigenstates of $\hat{x}^2 + \hat{y}^2$ [4], or a continuous one $\{|x>\}$, composed of eigenstates of, say, $\hat{x}$ [3].

To quantize Φ [3], start with a usual classical commuting field, expanded into normal modes with coefficients a and a^*. Upon usual field quantization, a and a^* become operators acting on a standard Fock space $\mathcal{F}$. To make the underlying space noncommutative, introduce (1) and apply the Weyl quantization procedure [5] to the exponentials $e^{i(k_x x + k_y y)}$. The result is

$$\Phi = \int\int \frac{dk_x dk_y}{2\pi\sqrt{2\omega_{\vec{k}}}}\left[\hat{a}_{k_x k_y} e^{i(\omega_{\vec{k}}t - k_x\hat{x} - k_y\hat{y})} + \hat{a}^\dagger_{k_x k_y} e^{-i(\omega_{\vec{k}}t - k_x\hat{x} - k_y\hat{y})}\right]. \qquad (3)$$

which means the following: Φ is a 'doubly'-quantum field operator, acting on a direct product of two Hilbert spaces, $\Phi : \mathcal{F}\otimes\mathcal{H} \to \mathcal{F}\otimes\mathcal{H}$. Physically, Φ creates (destroys), via $\hat{a}^\dagger_{k_x k_y}$ ($\hat{a}_{k_x k_y}$), an excitation represented by a "plane wave" $e^{i(\omega_{\vec{k}}t - k_x\hat{x} - k_y\hat{y})}$. The nature of such an excitation will be discussed now.

One could work with Φ as an operator ready to act on both $\mathcal{F}$ and $\mathcal{H}$. It is however simpler to saturate its action on $\mathcal{H}$, working with expectation values $< x'|\Phi|x >$: $\mathcal{F} \to \mathcal{F}$. It is at this point, of eliminating noncommutativity, that bilocality appears. To see that, consider the family $\{|x>\}$ of eigenstates of $\hat{x}$: $\hat{x}|x> = x|x>$, $\hat{y}|x> = -i\theta\frac{\partial}{\partial x}|x>$. A simple but key equation is

$$< x'|e^{i(k_x\hat{x} + k_y\hat{y})}|x> = e^{ik_x(x + k_y\theta/2)}\delta(x' - x - k_y\theta) = e^{ik_x\frac{x+x'}{2}}\delta(x' - x - k_y\theta). \qquad (4)$$

This is a bilocal expression, and we already see that its span along the x axis, $(x' - x)$, is proportional to the momentum along the conjugate y direction, i.e. $(x' - x) = \theta k_y$. In general, for n pairs of NC directions, one can keep only one coordinate out of every pair; commutativity is gained on the reduced space, at the expenses of strict locality. Using (3,4), one sees that

$$< x'|\Phi|x > = \int \frac{dk_x}{2\pi\sqrt{2\omega_{k_x,k_y}}}\left[\hat{a}_{k_x,k_y} e^{i(\omega_{\vec{k}}t - k_x\frac{x+x'}{2})} + \hat{a}^\dagger_{k_x,-k_y} e^{-i(\omega_{\vec{k}}t + k_x\frac{x+x'}{2}}\right] \qquad (5)$$

where $k_y = (x' - x)/\theta$. Thus, Φ annihilates a rod of (arbitrary) momentum k_x and (fixed) length θk_y, and creates a rod of momentum k_x and length

$-\theta k_y$. Due to (1), one degree of freedom apparently disappears from (5). Its presence shows up only through the modified dispersion relation

$$\omega_{(k_x, k_y=\frac{x'-x}{\theta})} = \sqrt{k_x^2 + \frac{(x'-x)^2}{\theta^2} + m^2}. \tag{6}$$

One notices the intrinsic IR/UV-dual character of the dipoles: both big momentum (UV) and big extension (IR) increase the energy. This second term reminds a string stretched between two separated D-branes.

Other bases can also be used for $\mathcal{H}$. For instance, the basis $\{|n>\}$, formed by the eigenvectors of $\hat{n} \sim x^2 + y^2$, leads to a discrete remnant space [4].

Correlators

Two-point correlation functions for such dipoles are the VEV of the product of two bilocal fields (taken on the vacuum, $|0\rangle$, of the Fock space $\mathcal{F}$):

$$\langle 0| < x_4|\Phi|x_3 >< x_2|\Phi|x_1 > |0\rangle = \int \frac{dk_x}{8\pi^2 \omega_{\vec{k}}} e^{ik_x[\frac{x_3+x_4}{2} - \frac{x_1+x_2}{2}]} \delta(x_4 - x_3 - x_2 + x_1). \tag{7}$$

Again, $k_y = (x'-x)/\theta$, $\omega_{\vec{k}} = \omega_{k_x,k_y}$ obeys (6), and there is no integral along k_y. If one compares (7) to the $(1+1)$-dimensional correlator of two commutative fields, $\langle 0|\phi(X_2)\phi(X_1)|0\rangle$, with $X_1 = (x_1+x_2)/2$ and $X_2 = (x_3+x_4)/2$, the differences are the $\frac{(x'-x)^2}{\theta^2}$ term in (6), and the delta function $\delta([x_4 - x_3] - [x_2 - x_1])$, which ensures that the length of the rod is conserved. Thus, our bilocal objects propagate in a $(1+1)-$dimensional space. The extra y direction is accounted for by their lenght, which contributes to the energy, and their orientation. Although we also call these rods dipoles, they do not necessarily have charges at their ends and they have extension in the absence of any background. Those rods may remind one about stretched open strings, or the double index representation of Yang-Mills theories.

Interactions

The quartic interaction term in (2) can be written as

$$\int dt Tr_{\mathcal{H}} V(\Phi) = \frac{g}{4!} \int dt \int_{x,a,b,c} < x|\Phi|a >< a|\Phi|b >< b|\Phi|c >< c|\Phi|x > . \tag{8}$$

To find the Feynman rules, we need the vacuum correlator (7), and a slight modification of the Dyson procedure. The basic 'vertex' for four-dipole scattering follows from

$$\langle -\vec{k}_3, -\vec{k}_4| : \int dt \int_{x,a,b,c} < x|\Phi|a >< a|\Phi|b >< b|\Phi|c >< c|\Phi|x >: |\vec{k}_1, \vec{k}_2\rangle. \tag{9}$$

$|\vec{k}_1, \vec{k}_2\rangle$ is a Fock space state with two quanta of momentum $\vec{k}_1$ and $\vec{k}_2$. The momenta $\vec{k}_{i,i=1,2,3,4}$ have each two components: $\vec{k}_i = (k_i, l_i)$. k_i is the momentum along x, whereas l_i represents the dipole extension along x (corresponding to the momentum along y) . Using Eq. (5) and integrating over x, a, b and c, one obtains the conservation laws $k_1 + k_2 + k_3 + k_4 = 0$ and $l_1 + l_2 + l_3 + l_4 = 0$. The final result differs from the four-point scattering vertex of $(2+1)$ commutative particles with momenta $\vec{k}_i = (k_i, l_i)$ only through the phase

$$e^{-\frac{i\theta}{2}\sum_{i<j}(k_i l_j - l_i k_j)}. \tag{10}$$

This is precisely the star-product modification of the usual Feynman rules. The phase (10) appears due to the bilocal nature of generic $< x'|\Phi|x >$'s.

By contracting various terms in (9), one obtains the one-loop corrections to the free rod propagator, together with the recipe for calculating loops. Again, the derivation is straightforward. The main point is that, in the end, one has to integrate over both the momentum and length of the dipole circulating in a loop. This $\frac{1}{2\pi}\int dk_{loop}\int dl_{loop}$ integration, together with the dispersion relation (6), brings back into play - especially as far as divergences are concerned - the y direction. It is easy to extend the above reasoning to $(2n+1)-$dimensions: unconstrained dipoles will propagate in a $(n+1)$-dimensional commutative space-time, with Feynman rules obtained as outlined above. Once the dipole lengths are interpreted as momenta in the conjugate directions, the rules are identical to those obtained long ago via star-product calculus.

IR/UV

We have derived directly from field theory the dipolar character of NC excitations; the momentum in the conjugate direction became the lenght of the dipole. A connection between UV and IR physics appeared naturally, and on a somehow more rigorous basis than in [6], for instance.

One can also view geometrically the differences between planar and non-planar loop diagrams, and the role of low momenta in nonplanar graphs. To illustrate this, consider $(4+1)$-dimensions, $t, \hat{x}, \hat{y}, \hat{z}, \hat{u}$, with $[\hat{x}, \hat{y}] = [\hat{z}, \hat{w}] = i\theta$. In the $\{|x, z >\}$ basis, one has a commutative space spanned by the axes x and z, on which dipoles with momentum $\vec{p} = (p_x, p_z)$ and length $\vec{l} = (l_x, l_z) = \theta(p_y, p_w)$ evolve. During the scattering, four such dipoles meet in a four-edged poligon of area $\mathcal{A}$ (figure 1a).

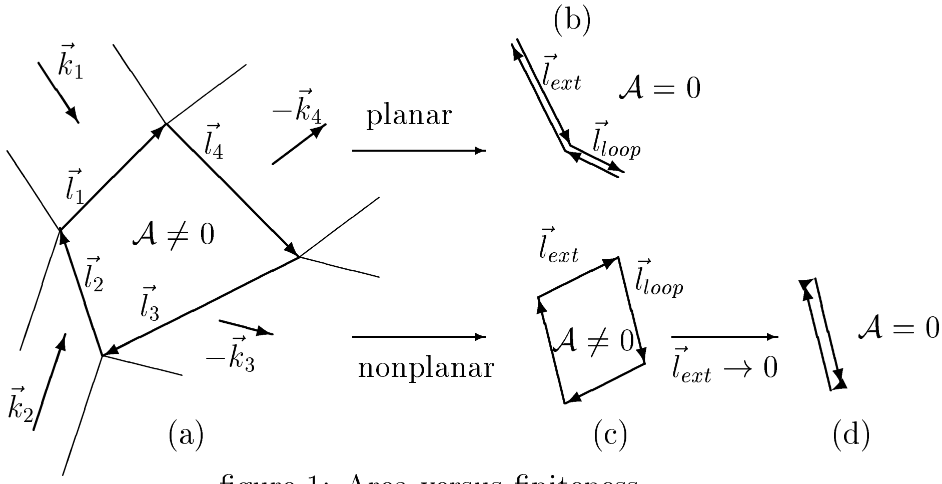

figure 1: Area versus finiteness

One has two possibilities for the one-loop correction to the propagator: planar and nonplanar. In the planar case, adjacent dipole fields are contracted. Momentum and length conservation enforce then the poligon to degenerate into a one-dimensional, zero-area object (figure 1b). UV divergences persist. In the nonplanar case, due to the nonadjacent contraction the area $\mathcal{A}$ does not go to zero (cf. figure 1c) unless the external dipole length vanishes (figure 1d). $\mathcal{A}\neq 0$ appears thus to be related to the disappearance of UV divergences. Actually, the true regulator is the phase (10). This is zero, i.e. ineffective, when $\mathcal{A}=0$ in *both* the $|x,z>$ and $|y,u>$ bases. That corresponds to zero external length *and* momentum in the dipole picture, which means that the resulting divergence is half IR ($\vec{p}_{ext}=0$) and half UV ($\vec{l}_{ext}=0$)! In Weyl space this is just the usual zero external momentum, say $p_\mu^{ext}=0$, and one speaks about an IR divergence. For dipoles the divergence comes from having zero vertex area $\mathcal{A}$ in any basis, and is half IR and half UV. NCFT appears to be somehow in between usual FT and string theory: when the interaction vertex is a point, UV infinities appear; when it opens up, as in string theory, amplitudes are finite.

Remarks

We saw that by dropping n coordinates, intuition is gained: the remaining space admits a notion of distance, although bilocal (and in some sense IR/UV dual) objects probe it. An important question is: how do the dimensionality and noncommutativity of space-time exactly depend on the regime in which we probe the theory? To start, we have a NC $(2n+1)-$dimensional theory. Then, at tree level (i.e. classical plus tree level interference effects), one has $D = n+1$ commuting directions. However, loop effects drive us back to $D = 2n+1$. At a scale $r \sim \sqrt{\theta}$, space is NC. For $r >> \sqrt{\theta}$ it is believed to be commutative. However, if r is the radius in the largest available commutative subspace, the IR/UV connection suggests a connection (duality?) between the $r >> \sqrt{\theta}$ and $r << \sqrt{\theta}$ regimes. A clarification of these issues is desirable.

One may also consider the case in which time is NC, e.g. $[\hat{t}, \hat{x}] \neq 0$. In a basis in which $\hat{t}$ is diagonal, $\{|t, \ldots>\}$, the elementary excitations become bilocal in time, $< t, \ldots|\Phi|t', \ldots >$. Their time-length contributes to the energy, $\omega = \sqrt{(t-t')^2/\theta^2 + k_x^2 + k_y^2 + m^2}$. Preliminay results indicate that, upon appropriate definition of the perturbation series, the theory *is* unitary, in agreement with [7] and in disagreement with [8].

Acknowledgments

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