Polarized Dirac fermions in de Sitter spacetime

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Abstract

The tetrad gauge invariant theory of the free Dirac field in two moving frames of the de Sitter spacetime is investigated pointing out the operators that commute with the Dirac one. These are the generators of the symmetry transformations corresponding to isometries that give rise to conserved quantities according to the Noether theorem. With their help the plane wave spinor solutions of the Dirac equation with given momentum and helicity are derived and the final form of the quantum Dirac field is established.

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1 Introduction

The recent astrophysical investigations showing that the expansion of the universe is accelerating [1] may increase the interest for the de Sitter spacetime which could represent the far future limit of the actual universe. On the other hand, the Dirac fermions (leptons and quarks) are the principal components of the matter because their gauge symmetries determine the main features of the physical picture. For these reasons, we believe that the study of the tetrad gauge invariant theory of the free Dirac field in de Sitter background may be important for understanding the influence of the external gravitational field minimally coupled with the fermion fields.

The Dirac equation in de Sitter spacetime (of radius $R = 1/\omega = \sqrt{3/\Lambda_c}$, produced by the cosmological constant Λ_c) has been studied in moving or static local charts (i.e. natural frames) suitable for separation of variables leading to significant analytical solutions [4, 5, 6]. The first spinor solutions on this background were obtained in a static central chart using the diagonal tetrad gauge in spherical coordinates [4]. Few years later, with a new method [7], spherical wave solutions of the Dirac equation were derived in the moving local chart $\{t, r, \theta, \phi\}$ associated to the Cartesian one $\{t, \vec{x}\}$ with the line element

$$ds^2 = dt^2 - e^{2\omega t} d\vec{x}^2 \,, \tag{1}$$

where a Cartesian tetrad gauge was considered [5]. Moreover, in [5] possible plane wave solutions in Cartesian coordinates were mentioned without writing them down explicitly. Since in these moving charts the operator $i\partial_t$ is no longer a Killing vector field, the quantum modes corresponding to all these particular solutions have no well-determined energies. Obviously, this is not an impediment but, in addition, there are some integration constants the physical meaning of which remains obscure [5]. Thus, actually we do not have yet a complete system of particular solutions that may be used for writing the general form of the quantum Dirac field in the de Sitter spacetime.

The main purpose of the present paper is to show that there are particular solutions suitable for expressing the canonically quantized [9] Dirac field in terms of creation and annihilation operators of fermions with well-defined physical properties. To this end we exploit the results of our previously constructed theory of external symmetry [10] which explains the relations among the geometric symmetries and the operators commuting with the Dirac one that have been written with the help of the Killing vectors some time ago [11]. In fact these operators are nothing other than the generators of the spinor representation of the universal covering group of the isometry group [10] and, therefore, they represent the main physical observables among which we can choose different sets of commuting operators defining quantum modes. This method is efficient especially in the case of the de Sitter spacetime where the high symmetry given by the $SO(4,1)$ isometry group [12, 13] offers the opportunity of a rich algebra of operators able to receive a physical meaning. Therefore, we interpret the generators of the subgroup $E(3) \subset SO(4,1)$ as the three-dimensional momentum and (orbital) angular momentum operators [14]. Then the corresponding generators of the spinor representation are the momentum and the total angular momentum operators. From this algebra we select the momentum components and, in addition, we construct an one-component Pauli-Lubanski (or helicity) operator [15], obtaining thus the set of commuting observables that defines quantum modes with given momentum and helicity. We show that the common eigenspinors of these operators are the desired plane wave solutions of the Dirac equation which can be easily normalized in the momentum scale. Moreover, we demonstrate that the system of these solutions is complete (in generalized sense).

This set is used for expanding the free Dirac field in terms of creation and annihilation operators of fermions characterized by momentum and helicity, pointing out that the canonical quantization requires us to adopt the standard anticommutation rules in momentum representation. In this way the conserved quantities predicted by the Noether theorem become the one-particle operators of the quantum field theory, among them the diagonal ones are the momentum, helicity and charge operators. Thus in our approach the second quantization can be done in canonical manner obtaining new results specific to the de Sitter geometry.

The definition of the physical observables in section III, the form of the normalized plane wave solutions of the Dirac equation as well as all the other results of sections IV and V are original. These are presented in natural units with $\hbar = c = 1$.

2 Gauge and external symmetry

In a curved spacetimes M the choice of the local charts, $\{x\}$, of coordinates x^μ $(\mu,\nu,... = 0,1,2,3)$ is important from the observer point of view. In addition, the tetrad gauge covariant theory of the fields with spin requires to explicitly use the tetrad fields, $e_{\hat\mu}(x)$ and $\hat e^{\hat\mu}(x)$, fixing the local frames and the corresponding coframes. These are labeled by the local indices, $\hat\mu,\hat\nu,... = 0,1,2,3$, and have the orthonormalization properties $e_{\hat\mu}\cdot e_{\hat\nu} = \eta_{\hat\mu\hat\nu}$, $\hat e^{\hat\mu}\cdot\hat e^{\hat\nu} = \eta^{\hat\mu\hat\nu}$ and $\hat e^{\hat\mu}\cdot e_{\hat\nu} = \delta^{\hat\mu}_{\hat\nu}$, with respect to the Minkowski metric $\eta =$diag$(1,-1,-1,-1)$. The 1-forms $d\hat x^{\hat\mu} = \hat e^{\hat\mu}_\nu dx^\nu$ allow one to write the line element

$$ds^2 = \eta_{\hat\mu\hat\nu}d\hat x^{\hat\mu}d\hat x^{\hat\nu} = g_{\mu\nu}(x)dx^\mu dx^\nu\,, \tag{2}$$

defining the metric tensor $g_{\mu\nu}$ (which raises or lowers the Greek indices while for the hated Greek ones we have to use the Minkowski metric). The derivatives in local frames are the vector fields $\hat{\partial}_{\hat{\nu}} = e^{\mu}_{\hat{\nu}}\partial_{\mu}$ which satisfy the commutation rules $[\hat{\partial}_{\hat{\mu}}, \hat{\partial}_{\hat{\nu}}] = C_{\hat{\mu}\hat{\nu}\cdot}^{\cdot\cdot\hat{\sigma}}\hat{\partial}_{\hat{\sigma}}$ giving the Cartan coefficients that help us to write the conecttion components in local frames.

Let ψ be a Dirac free field of mass m, defined on the space domain D, and $\overline{\psi} = \psi^{+}\gamma^{0}$ its Dirac adjoint. The tetrad gauge invariant action of the Dirac field minimally coupled with the gravitational field is

$$\mathcal{S}[e, \psi] = \int d^4x\sqrt{g}\left\{\frac{i}{2}[\overline{\psi}\gamma^{\hat{\alpha}}D_{\hat{\alpha}}\psi - (\overline{D_{\hat{\alpha}}\psi})\gamma^{\hat{\alpha}}\psi] - m\overline{\psi}\psi\right\} \tag{3}$$

where $g = |\det(g_{\mu\nu})|$ and the Dirac matrices, $\gamma^{\hat{\alpha}}$, satisfy $\{\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\} = 2\eta^{\hat{\alpha}\hat{\beta}}$. The covariant derivatives in local frames, $D_{\hat{\alpha}} = e^{\mu}_{\hat{\alpha}}D_{\mu} = \hat{\partial}_{\hat{\alpha}} + \hat{\Gamma}_{\hat{\alpha}}$, are expressed in terms of the spin connections $\hat{\Gamma}_{\hat{\mu}} = \hat{\Gamma}_{\hat{\mu}\hat{\nu}\hat{\lambda}}S^{\hat{\nu}\hat{\lambda}} = \frac{i}{4}(C_{\hat{\mu}\hat{\nu}\hat{\lambda}} - C_{\hat{\mu}\hat{\lambda}\hat{\nu}} - C_{\hat{\nu}\hat{\lambda}\hat{\mu}})S^{\hat{\nu}\hat{\lambda}}$ given by the basis-generators in covariant parametrization, $S^{\hat{\alpha}\hat{\beta}} = \frac{i}{4}[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]$, of the usual spinor representation $\rho \sim (1/2, 0)\oplus(0, 1/2)$ of the $SL(2, \mathbb{C})$ group [15, 16] (i.e. the universal covering group of the Lorentz group, $L^{\uparrow}_{+}$, that is the gauge group of the metric η). The Dirac operator of the equation $E_D\psi = m\psi$ derived from the action (3) reads $E_D = i\gamma^{\hat{\alpha}}D_{\hat{\alpha}}$. In other respects, from the conservation of the electric charge one deduces that when $e^{0}_{i} = 0$ $(i, j, ... = 1, 2, 3)$ the time-independent relativistic scalar product of two spinors [17],

$$\langle\psi, \psi'\rangle = \int_D d^3x\, \mu(x)\overline{\psi}(x)\gamma^{0}\psi'(x)\,, \tag{4}$$

has the weight function $\mu = \sqrt{g}\, e^{0}_{0}$.

The action (3) is gauge invariant in the sense that it remains unchanged when one performs a gauge transformation $\psi(x) \to \psi'(x) = \rho[A(x)]\psi(x)$ and $e_{\hat{\alpha}}(x) \to e'_{\hat{\alpha}}(x) = \Lambda^{\cdot\hat{\beta}}_{\hat{\alpha}\cdot}[A(x)]e_{\hat{\beta}}(x)$ produced by $A(x) \in SL(2, \mathbb{C})$ and $\Lambda[A(x)] \in L^{\uparrow}_{+}$. Based on this symmetry, we have defined the group of external symmetry, $S(M)$, corresponding to the isometry group $I(M)$. The transformations of $S(M)$ are isometries of $I(M)$, $x \to x' = \phi_{\xi}(x)$ (depending on the parameters ξ^{a}, $a = 1, 2, ...n$), combined with appropriate gauge transformations in such a manner to preserve the tetrad gauge. In a fixed gauge, one associates to each isometry ϕ_{ξ} the section $A_{\xi}(x) \in SL(2, \mathbb{C})$ defined by

$$\Lambda^{\hat{\alpha}\cdot}_{\cdot\hat{\beta}}[A_{\xi}(x)] = \hat{e}^{\hat{\alpha}}_{\mu}[\phi_{\xi}(x)]\frac{\partial\phi^{\mu}_{\xi}(x)}{\partial x^{\nu}}\, e^{\nu}_{\hat{\beta}}(x) \tag{5}$$

with the supplementary condition $A_{\xi=0}(x) = 1 \in SL(2,C)$. Then the transformations of the group $S(M)$ are

$$(A_\xi, \phi_\xi): \qquad \begin{array}{rcrcl} x & \to & x' & = & \phi_\xi(x) \\ e(x) & \to & e'(x') & = & e[\phi_\xi(x)] \\ \hat{e}(x) & \to & \hat{e}'(x') & = & \hat{e}[\phi_\xi(x)] \\ \psi(x) & \to & \psi'(x') & = & \rho[A_\xi(x)]\psi(x)\,. \end{array} \tag{6}$$

In [10] we presented arguments that $S(M)$ is the universal covering group of $I(M)$ remarking that the representation defined by the last of the equations (6) is not an usual linear one of $S(M)$. In fact this is *induced* by the representation ρ of the group $SL(2,\mathbb{C})$ which, in general, differs from $S(M)$. For this reason we say that ψ transforms according to the spinor representation of $S(M)$ induced by ρ.

The transformations (6) leave invariant the form of the operator E_D in local frames. Consequently, each Killing vector, $k_a = (\partial_{\xi^a}\phi_\xi)_{|\xi=0}$, defines a basis-generator of the spinor representation [11, 10],

$$X_a = -ik_a^\mu D_\mu + \frac{1}{2}\, k_{a\,\mu;\nu}\, e^\mu_{\hat{\alpha}}\, e^\nu_{\hat{\beta}}\, S^{\hat{\alpha}\hat{\beta}}\,, \tag{7}$$

which *commutes* with E_D (the notation $_{;\nu}$ stands for the usual covariant derivatives). We must specify that this important result was obtained for the Dirac field in [11] without to take into account the symmetry transformations. In [10] we have shown that the generators (7) satisfy the commutation relations $[X_a, X_b] = ic_{abc}X_c$, $(a, b, c = 1, 2, ...n)$, given by the structure constants of the isometry group $I(M)$. On the other hand, each generator can be split in an orbital and spin part as $X_a = L_a + S_a$, where the orbital terms

$$L_a = -ik_a^\mu(x)\partial_\mu \tag{8}$$

are the basis-generators of the natural representation of $I(M)$ carried by the space of the scalar functions over M. The spin terms

$$S_a(x) = \frac{1}{2}\,\Omega_a^{\hat{\alpha}\hat{\beta}}(x) S_{\hat{\alpha}\hat{\beta}} \tag{9}$$

are defined with the help of the functions

$$\Omega_a^{\hat{\alpha}\hat{\beta}} = \left(\hat{e}^{\hat{\alpha}}_\mu\, \partial_\nu k_a^\mu + k_a^\mu \partial_\mu \hat{e}^{\hat{\alpha}}_\nu\right) e^\nu_{\hat{\lambda}} \eta^{\hat{\lambda}\hat{\beta}} \tag{10}$$

that are antisymmetric if and only if k_a is a Killing vector. Thus we see that the spin terms of the generators X_a generally depend on x and, therefore, they do not commute with the orbital terms. When L_a and S_a commute between themselves we say that the Dirac field transforms *manifestly* covariant under the symmetry transformations parametrized by ξ^a.

Our theory of external symmetry offers us the framework we need to calculate the conserved quantities predicted by the Noether theorem. Starting with the infinitesimal transformations of the one-parameter subgroup of $S(M)$ generated by X_a, we find that there exists the conserved current $\Theta^{\mu}[X_a]$ which satisfies $\Theta^{\mu}[X_a]_{;\mu} = 0$. For the action (3) this is $\Theta^{\mu}[X_a] = -\tilde{T}^{\mu\,\cdot}_{\cdot\,\nu} k_a^{\nu} + \frac{1}{4}\,\overline{\psi}\{\gamma^{\hat{\alpha}}, S^{\hat{\beta}\hat{\gamma}}\}\psi\, e^{\mu}_{\hat{\alpha}}\,\Omega_{a\,\hat{\beta}\hat{\gamma}}$ where $\tilde{T}^{\mu\,\cdot}_{\cdot\,\nu} = \frac{i}{2}\left[\overline{\psi}\gamma^{\hat{\alpha}}e^{\mu}_{\hat{\alpha}}\partial_{\nu}\psi - (\overline{\partial_{\nu}\psi})\gamma^{\hat{\alpha}}e^{\mu}_{\hat{\alpha}}\psi\right]$ is a notation for a part of the stress-energy tensor of the Dirac field [12, 17]. Finally, it is clear that the corresponding conserved quantity is the real number

$$\int_D d^3x\sqrt{g}\,\Theta^0[X_a] = \frac{1}{2}\left[\langle\psi, X_a\psi\rangle + \langle X_a\psi, \psi\rangle\right]\,. \tag{11}$$

We note that it is premature to interpret this formula as an expectation value or to speak about Hermitian conjugation of the operators X_a with respect to the scalar product (4), before specifying the boundary conditions on D. What is important here is that this result is useful in quantization giving directly the one-particle operators of the quantum field theory.

3 Observables in de Sitter spacetime

Let us consider now M be the de Sitter spacetime, defined as the hyperboloid of radius R in the five-dimensional flat spacetime M^5 of coordinates Z^A ($A, B, ... = 0, 1, 2, 3, 5$) and metric $\eta^5 = \mathrm{diag}(1, -1, -1, -1, -1)$ [12]. The hyperboloid equation $\eta^5_{AB}Z^AZ^B = -R^2$ defines M as the homogeneous space of the pseudo-orthogonal group $SO(4, 1)$ which is at the same time the gauge group of the metric η^5 and the isometry group, $I(M)$, of the de Sitter spacetime. For this reason it is convenient to use the covariant real parameters $\xi^{AB} = -\xi^{BA}$ since in this case the orbital basis-generators of the representation of $SO(4, 1)$, carried by the space of the scalar functions over M^5, have the standard form

$$L^5_{AB} = i\left[\eta^5_{AC}Z^C\partial_B - \eta^5_{BC}Z^C\partial_A\right]. \tag{12}$$

They will give us directly the orbital basis-generators $L_{(AB)}$ of the scalar representations of $I(M)$. Indeed, starting with the functions $Z^A(x)$ that solve the hyperboloid equation in the chart $\{x\}$, one can write down the operators (12) in the form (8), finding thus the generators $L_{(AB)}$ and implicitly the components $k^{\mu}_{(AB)}(x)$ of the Killing vectors associated to the parameters ξ^{AB} [10]. Furthermore, one has to calculate the spin parts $S_{(AB)}$, according to (9) and (10), arriving to the final form of the basis-generators $X_{(AB)} = L_{(AB)} + S_{(AB)}$ of the spinor representation of $S(M)$ induced by ρ.

In the de Sitter spacetime there are many static or moving charts of physical interest. Among the moving ones a special role plays the chart $\{t_c, \vec{x}\}$ with the conformal time t_c and Cartesian suitable space coordinates x^i defined by $Z^0 = -(2\omega^2 t_c)^{-1}[1-\omega^2({t_c}^2 - r^2)]$, $Z^5 = -(2\omega^2 t_c)^{-1}[1+\omega^2({t_c}^2 - r^2)]$ and $Z^i = -(\omega t_c)^{-1} x^i$, with $r = |\vec{x}|$. Even if this chart covers only a half of the manifold M, for $t_c \in (-\infty, 0)$ and $\vec{x} \in D \equiv \mathbb{R}^3$, it has the advantage of a simple conformal flat line element [17],

$$ds^2 = \frac{1}{\omega^2 {t_c}^2}\left(d{t_c}^2 - d\vec{x}^2\right) . \tag{13}$$

Moreover, the conformal time t_c is related through $\omega t_c = -e^{-\omega t}$ to the proper time $t \in (-\infty, \infty)$ of an observer at $\vec{x} = 0$ in the chart $\{t, \vec{x}\}$ with the line element (1). In what follows we study the Dirac field in the chart $\{t, \vec{x}\}$ using the conformal time as a helpful auxiliary ingredient. The form of the line element (13) allows one to choose the simple Cartesian gauge with the non-vanishing tetrad components [5]

$$e^0_0 = -\omega t_c\,, \quad e^i_j = -\delta^i_j\,\omega t_c\,, \quad \hat{e}^0_0 = -\frac{1}{\omega t_c}\,, \quad \hat{e}^i_j = -\delta^i_j\,\frac{1}{\omega t_c}\,. \tag{14}$$

In this gauge the Dirac operator reads

$$\begin{aligned} E_D &= -i\omega t_c\left(\gamma^0\partial_{t_c} + \gamma^i\partial_i\right) + \frac{3i\omega}{2}\gamma^0 \\ &= i\gamma^0\partial_t + ie^{-\omega t}\gamma^i\partial_i + \frac{3i\omega}{2}\gamma^0 \end{aligned} \tag{15}$$

and the weight function of the scalar product (4) is

$$\mu = (-\omega t_c)^{-3} = e^{3\omega t}\,. \tag{16}$$

The next step is to calculate the basis-generators $X_{(AB)}$ of the spinor representation of $S(M)$ in this gauge since these are the main operators that commute with E_D. The group $SO(4,1)$ includes the subgroup $E(3) = T(3)Ⓢ SO(3)$ which is just the isometry group of the 3-dimensional Euclidean space of our moving charts, $\{t_c, \vec{x}\}$ and $\{t, \vec{x}\}$, formed by $\mathbb{R}^3$ translations, $x^i \to x^i + a^i$, and proper rotations, $x^i \to R^i_{\cdot j} x^j$ with $R \in SO(3)$ [15]. Therefore, the basis-generators of its universal covering group, $\tilde{E}(3) = T(3)Ⓢ SU(2) \subset S(M)$, can be interpreted as the components of the momentum, $\vec{P}$, and total angular momentum, $\vec{J}$, operators. The problem of the Hamiltonian operator seems to be more complicated, but we know that in the mentioned static central charts with the static time t_s this is $H = \omega X_{(05)} = i\partial_{t_s}$ [10]. Thus the Hamiltonian operator and the components of the momentum and total angular momentum operators (P^i and $J^i = \varepsilon_{ijk} J_{jk}/2$ respectively) can be identified as being the following basis-generators of $S(M)$

$$H \equiv \omega X_{(05)} = -i\omega(t_c \partial_{t_c} + x^i \partial_i) \tag{17}$$

$$P^i \equiv \omega \left(X_{(5i)} - X_{(0i)} \right) = -i\partial_i \tag{18}$$

$$J_{ij} \equiv X_{(ij)} = -i(x^i \partial_j - x^j \partial_i) + S_{ij} \tag{19}$$

after which one remains with the three basis-generators

$$N^i \equiv X_{(5i)} + X_{(0i)} = \omega({t_c}^2 - r^2)P^i + 2x^i H + 2\omega(S_{i0} t_c + S_{ij} x^j)\,, \tag{20}$$

which do not have an immediate physical significance. The $SO(4,1)$ transformations corresponding to these basis-generators and the associated isometries of the chart $\{t_c, \vec{x}\}$ are briefly presented in the Appendix A.

In the other local chart, $\{t, \vec{x}\}$, we have the same operators $\vec{P}$ and $\vec{J} = \vec{L} + \vec{S}$ (with $\vec{L} = \vec{x} \times \vec{P}$) whose components are the $\tilde{E}(3)$ generators, while the Hamiltonian operator takes the form

$$H = i\partial_t + \omega\, \vec{x} \cdot \vec{P}\,, \tag{21}$$

where the second term, due to the external gravitational field, leads to the commutation rules

$$[H,\, P^i] = i\omega P^i\,. \tag{22}$$

We observe that in this chart the operators $K^i \equiv X_{(0i)}$ are the analogous of the basis-generators of the Lorentz boosts of $SL(2, \mathbb{C})$ since in the limit of $\omega \to 0$, when (1) equals the Minkowski line element, the operators $H = P^0$, P^i, J^i and K^i become the generators of the spinor representation of the group $T(4)\circledS SL(2,\mathbb{C})$ (i.e. the universal covering group of the Poincaré group [15, 16]).

In both the charts we used here the generators (17)-(20) are self-adjoint with respect to the scalar product (4) with the weight function (16) if we consider the usual boundary conditions on $D \equiv \mathbb{R}^3$. Therefore, for any generator X we have $\langle X\psi, \psi'\rangle = \langle \psi, X\psi'\rangle$ if (and only if) ψ and ψ' are solutions of the Dirac equation which behave as tempered distributions or square integrable spinors with respect to the scalar product (4). Moreover, all these generators commute with the Dirac operator E_D. If, in addition, we take into account the algebra freely generated by them, then we get a large collection of observables among which we can choose suitable sets of commuting operators defining the fermion quantum modes at the level of the relativistic quantum mechanics.

4 Plane wave solutions and quantisation

As suggested in [5], the plane wave solutions of the Dirac equation with $m \neq 0$ must be eigenspinors of the momentum operators P^i corresponding to the eigenvalues p^i, with the same time modulation as the spherical waves. Therefore, we have to look for particular solutions in the chart $\{t_c, \vec{x}\}$ involving either positive or negative frequency plane waves. Bearing in mind that these must be related among themselves through the charge conjugation, we assume that, in the standard representation of the Dirac matrices (with diagonal γ^0 [16]), they have the form

$$\psi^{(+)}_{\vec{p}} = \left(\begin{array}{c} f^+(t_c)\, \alpha(\vec{p}) \\ g^+(t_c)\, \dfrac{\vec{\sigma}\cdot\vec{p}}{p}\, \alpha(\vec{p}) \end{array} \right) e^{i\vec{p}\cdot\vec{x}} \tag{23}$$

$$\psi^{(-)}_{\vec{p}} = \left(\begin{array}{c} g^-(t_c)\, \dfrac{\vec{\sigma}\cdot\vec{p}}{p}\, \beta(\vec{p}) \\ f^-(t_c)\, \beta(\vec{p}) \end{array} \right) e^{-i\vec{p}\cdot\vec{x}} \tag{24}$$

where $p = |\vec{p}|$, σ_i denotes the Pauli matrices while α and β are arbitrary Pauli spinors depending on $\vec{p}$. Replacing these spinors in the Dirac equation given by (15) and denoting $k = m/\omega$ and $\nu_\pm = \frac{1}{2} \pm ik$, we find equations of the form (46) whose solutions can be written in terms of Hankel functions as

$$f^+ = (-f^-)^* = C{t_c}^2 e^{\pi k/2} H^{(1)}_{\nu_-}(-pt_c) \tag{25}$$
$$g^+ = (-g^-)^* = C{t_c}^2 e^{-\pi k/2} H^{(1)}_{\nu_+}(-pt_c)\,. \tag{26}$$

The integration constant C will be calculated from the ortonormalization condition in the momentum scale.

The plane wave solutions are determined up to the significance of the Pauli spinors α and β. For $\vec{p} \neq 0$ these can be treated as in the flat case [9, 16] since, in the tetrad gauge (14), the spaces of these spinors carry unitary linear representations of the $\tilde{E}(3)$ group. Indeed, a transformation (6) produced by $(A, \phi_{A,\vec{a}}) \in \tilde{E}(3) \subset S(M)$ where $A \in SU(2)$ and $\vec{a} \in \mathbb{R}^3$ involves the usual linear isometry of $E(3)$, $x^i \to x'^i = \phi^i_{A,\vec{a}}(\vec{x}) \equiv \Lambda^{i}_{\cdot j}(A)x^j + a^i$ with $\Lambda(A) \in SO(3)$, and the global transformation $\psi(t,\vec{x}) \to \psi'(t,\vec{x}') = \rho(A)\psi(t,\vec{x})$. Consequently, the Pauli spinors transform according to the unitary (linear) representation

$$\alpha(\vec{p}) \to e^{-i\vec{a}\cdot\vec{p}} A\, \alpha[\Lambda(A)^{-1}\vec{p}] \tag{27}$$

(and similarly for β) that preserves orthogonality. This means that any pair of orthogonal spinors $\tilde{\xi}_\sigma(\vec{p})$ with polarizations $\sigma = \pm 1/2$ (obeying $\tilde{\xi}^+_\sigma \tilde{\xi}_{\sigma'} = \delta_{\sigma\sigma'}$) represents a good basis in the space of Pauli spinors

$$\alpha(\vec{p}) = \sum_\sigma \tilde{\xi}_\sigma(\vec{p}) a(\vec{p},\sigma) \tag{28}$$

whose components, $a(\vec{p},\sigma)$, are the particle wave functions in momentum representation. According to the standard interpretation of the negative frequency terms [9, 16], the corresponding basis of the space of β spinors is defined by

$$\beta(\vec{p}) = \sum_\sigma \tilde{\eta}_\sigma(\vec{p})[b(\vec{p},\sigma)]^*\,, \quad \tilde{\eta}_\sigma(\vec{p}) = i\sigma_2[\tilde{\xi}_\sigma(\vec{p})]^* \tag{29}$$

where $b(\vec{p},\sigma)$ are the antiparticle wave functions. It remains to choose a specific basis, using supplementary physical assumptions. Since it is not sure that the so called spin basis [9] can be correctly defined in the de Sitter geometry, we prefer the *helicity* basis. This is formed by the orthogonal Pauli spinors of helicity $\lambda = \pm 1/2$ which fulfill

$$\vec{\sigma}\cdot\vec{p}\,\tilde{\xi}_\lambda(\vec{p}) = 2p\lambda\,\tilde{\xi}_\lambda(\vec{p})\,, \quad \vec{\sigma}\cdot\vec{p}\,\tilde{\eta}_\lambda(\vec{p}) = -2p\lambda\,\tilde{\eta}_\lambda(\vec{p})\,. \tag{30}$$

The desired particular solutions of the Dirac equation with $m \neq 0$ result from our starting formulas (23) and (24) where we insert the functions (25) and (26) and the spinors (28) and (29) written in the helicity basis (30). It remains to calculate the normalization constant C with respect to the scalar product (4) with the weight function (16). After a few manipulations, in the chart $\{t, \vec{x}\}$, it turns out that the final form of the fundamental spinor solutions of positive and negative frequencies with momentum $\vec{p}$ and helicity λ is

$$U_{\vec{p},\lambda}(t,\vec{x}) = iN \left(\begin{array}{c} \frac{1}{2}\, e^{\pi k/2} H^{(1)}_{\nu_-}(qe^{-\omega t})\, \tilde{\xi}_\lambda(\vec{p}) \\ \lambda\, e^{-\pi k/2} H^{(1)}_{\nu_+}(qe^{-\omega t})\, \tilde{\xi}_\lambda(\vec{p}) \end{array} \right) e^{i\vec{p}\cdot\vec{x}-2\omega t} \tag{31}$$

$$V_{\vec{p},\lambda}(t,\vec{x}) = iN \left(\begin{array}{c} -\lambda\, e^{-\pi k/2} H^{(2)}_{\nu_-}(qe^{-\omega t})\, \tilde{\eta}_\lambda(\vec{p}) \\ \frac{1}{2}\, e^{\pi k/2} H^{(2)}_{\nu_+}(qe^{-\omega t})\, \tilde{\eta}_\lambda(\vec{p}) \end{array} \right) e^{-i\vec{p}\cdot\vec{x}-2\omega t}\,, \tag{32}$$

where we introduced the new parameter $q = p/\omega$ and

$$N = \frac{1}{(2\pi)^{3/2}}\sqrt{\pi q}\,. \tag{33}$$

Using (45) and (47), it is not hard to verify that these spinors are charge-conjugated to each other,

$$V_{\vec{p},\lambda} = (U_{\vec{p},\lambda})^c = \mathcal{C}(\overline{U}_{\vec{p},\lambda})^T\,, \quad \mathcal{C} = i\gamma^2\gamma^0\,, \tag{34}$$

satisfy the ortonormalization relations

$$\langle U_{\vec{p},\lambda}, U_{\vec{p}\,',\lambda'}\rangle = \langle V_{\vec{p},\lambda}, V_{\vec{p}\,',\lambda'}\rangle = \delta_{\lambda\lambda'}\delta^3(\vec{p}-\vec{p}\,')\,, \tag{35}$$

$$\langle U_{\vec{p},\lambda}, V_{\vec{p}\,',\lambda'}\rangle = \langle V_{\vec{p},\lambda}, U_{\vec{p}\,',\lambda'}\rangle = 0\,, \tag{36}$$

and represent a *complete* system of solutions in the sense that

$$\int d^3p \sum_\lambda \left[U_{\vec{p},\lambda}(t,\vec{x})U^+_{\vec{p},\lambda}(t,\vec{x}\,') + V_{\vec{p},\lambda}(t,\vec{x})V^+_{\vec{p},\lambda}(t,\vec{x}\,')\right] = e^{-3\omega t}\delta^3(\vec{x}-\vec{x}\,')\,. \tag{37}$$

Let us observe that the factor $e^{-3\omega t}$ is exactly the quantity necessary to compensate the weight function (16).

The quantization can be done considering that the wave functions in momentum representation, $a(\vec{p},\lambda)$ and $b(\vec{p},\lambda)$, become field operators (so that $b^* \to b^\dagger$) [9]. Then the quantum field which satisfies the Dirac equation with $m \neq 0$ in the chart $\{t, \vec{x}\}$ reads

$$\begin{aligned} \psi(t,\vec{x}) &= \psi^{(+)}(t,\vec{x}) + \psi^{(-)}(t,\vec{x}) \\ &= \int d^3p \sum_{\lambda} \left[U_{\vec{p},\lambda}(x) a(\vec{p},\lambda) + V_{\vec{p},\lambda}(x) b^\dagger(\vec{p},\lambda) \right] . \end{aligned} \tag{38}$$

We assume that the particle (a, $a^\dagger$) and antiparticle (b, $b^\dagger$) operators must fulfill the standard anticommutation relations in the momentum representation, from which the non-vanishing ones are

$$\{a(\vec{p},\lambda), a^\dagger(\vec{p}\,',\lambda')\} = \{b(\vec{p},\lambda), b^\dagger(\vec{p}\,',\lambda')\} = \delta_{\lambda\lambda'}\delta^3(\vec{p}-\vec{p}\,') , \tag{39}$$

since then the equal-time anticommutator takes the *canonical* form

$$\{\psi(t,\vec{x}), \overline{\psi}(t,\vec{x}\,')\} = e^{-3\omega t}\gamma^0\delta^3(\vec{x}-\vec{x}\,') , \tag{40}$$

as it results from (37). In general, the partial anticommutator functions,

$$\tilde{S}^{(\pm)}(t,t',\vec{x}-\vec{x}\,') = i\{\psi^{(\pm)}(t,\vec{x}), \overline{\psi}^{(\pm)}(t',\vec{x}\,')\} , \tag{41}$$

and the total one $\tilde{S} = \tilde{S}^{(+)} + \tilde{S}^{(-)}$ are rather complicated since for $t \neq t'$ we have no more identities like (47) which should simplify their time-dependent parts. In any event, these are solutions of the Dirac equation in both their sets of coordinates and help one to write the Green functions in usual manner. For example, from the standard definition of the Feynman propagator [9],

$$\tilde{S}_F(t,t',\vec{x}-\vec{x}\,') = i\,\langle 0|\,T[\psi(x)\overline{\psi}(x')]\,|0\rangle \tag{42}$$

$$= \theta(t-t')\tilde{S}^{(+)}(t,t',\vec{x}-\vec{x}\,') - \theta(t'-t)\tilde{S}^{(-)}(t,t',\vec{x}-\vec{x}\,') , \tag{43}$$

we find that

$$[E_D(x) - m]\tilde{S}_F(t,t',\vec{x}-\vec{x}\,') = -e^{-3\omega t}\delta^4(x-x') . \tag{44}$$

5 Concluding remarks

We have derived here a complete system of normalized plane wave solutions of the Dirac equation in the chart with the line element (1) of the de Sitter spacetime. These describe the quantum modes of polarized free fermions (or antifermions), determined by the complete set of commuting operators $\{E_D, \vec{S}^2, P^i, W\}$. A crucial point was the choice of the Cartesian gauge [5] in which the Dirac field transforms manifestly covariant under the $\tilde{E}(3)$ subgroup since in these conditions one can perform the second quantization in canonical way as in special relativity. We recall that in the static central charts $\{t_s, \vec{x}_s\}$ there is another appropriate Cartesian gauge where the Dirac field transforms manifestly covariant under the subgroup $T(1)_{t_s} \otimes SU(2) \subset S(M)$ (involving the time translations generated by $H = i\partial_{t_s}$) [10]. Then the separation of variables can be done as in the central problems of special relativity, leading to common eigenspinors of the complete set $\{E_D, \vec{S}^2, H, \vec{J}^2, J_3, \mathcal{K}\}$ [6] which includes the usual spin-orbit operator $\mathcal{K}$ [16]. This method allowed us to obtain the solutions presented in [6] as well as the normalized energy eigenspinors of the Dirac field in the anti-de Sitter spacetime [18]. Thus we draw the conclusion that one can reproduce similar conjectures as in the Minkowski flat spacetime if one exploits the manifest covariance with respect to a suitable subgroup of $S(M)$. All our examples [10] indicate that for each local chart there exists a specific subgroup of $S(M)$ the spinor representation of which can be brought in covariant form by an adequate tetrad gauge fixing. Obviously, this approach requires one to solve new problems from the pure mathematical ones up to those regarding the physical interpretation.

A Some properties of Hankel functions

According to the general properties of the Hankel functions [20], we deduce that those used here, $H^{(1,2)}_{\nu_\pm}(z)$, with $\nu_\pm = \frac{1}{2} \pm ik$ and $z \in \mathbb{R}$, are related among themselves through

$$[H^{(1,2)}_{\nu_\pm}(z)]^* = H^{(2,1)}_{\nu_\mp}(z)\,, \tag{45}$$

satisfy the equations

$$\left(\frac{d}{dz} + \frac{\nu_\pm}{z}\right) H^{(1)}_{\nu_\pm}(z) = i e^{\pm\pi k} H^{(1)}_{\nu_\mp}(z) \tag{46}$$

and the identities

$$e^{\pm\pi k} H^{(1)}_{\nu_\mp}(z) H^{(2)}_{\nu_\pm}(z) + e^{\mp\pi k} H^{(1)}_{\nu_\pm}(z) H^{(2)}_{\nu_\mp}(z) = \frac{4}{\pi z}\,. \tag{47}$$

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