Generalizing the q-symmetrized Harper-equation

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Abstract

The q-symmetrized Harper-equation [P. B. Wiegmann and V. A. Zabrodin, Phys. Rev. Lett. **72**, 1890 (1994)] is generalized by accounting for arbitrary values of the anisotropy parameter Δ. This parameter discriminates between metallic ($\Delta < 1$) and insulator ($\Delta > 1$) phases. Assuming that the wavefunction is described in terms of Laurent series, we succeeded to establish reasonable extrapolations of energy polynomials towards continuous values of the commensurability parameter, now for arbitrary Δ-values.

1 Introduction

The q-symmetrized Harper-equation

$$i\left(\frac{1}{z}+qz\right)\psi\left(qz\right)-i\left(\frac{z}{q}+\frac{1}{z}\right)\psi\left(q^{-1}z\right)=E\psi\left(z\right)\ , \tag{1}$$

has received much interest [1, 2, 3]. This equation serves to the middle band description of Bloch electrons on a $2D$ lattice threaded by a transversal and homogeneous magnetic field $\overrightarrow{B}=(0,0,B)$. One has

$$q=\exp\left(i\hbar^*/2\right)\ , \tag{2}$$

and $\hbar^*=2\pi\beta$, where β is a commesurability parameter expressing the number of flux quanta per unit cell. Accordingly $\beta=\Phi/\Phi_0$, where $\Phi=BS_\perp$, $S_\perp=a^2$ and $\Phi_0=h/e$. Here one deals with a square lattice with the spacing "a", but other kinds of lattices, like triangular or hexagonal ones, can also be considered. We shall also choose β-values like $\beta=P/Q$, where P and Q are mutually prime integers. In this case the q-parameter becomes a root of unity like $q^{2Q}=1$. Under such conditions proofs have been given that Eq. (1) exhibits the symmetry of the quantum group $sl_q\left(2\right)$ [1]. However, there is the possibility to account for a further parameter, namely the anisotropy parameter Δ, which characterizes from the very beginning the original form of the (dimensionless) energy dispersion law for a $2D$ square lattice:

$$E_{disp}\left(\overrightarrow{k}\right) = \cos\theta_1 + \Delta\cos\theta_2 \ , \tag{3}$$

where $\theta_j = k_j\ a\ (j = 1, 2)$ are the pertinent Brillouin phases. The anisotropy parameter referred to above discriminates between insulator ($\Delta > 1$) and metallic ($\Delta < 1$) phases [4]. This amounts to apply the minimal substitution (see also Ref. [5]) to Eq. (3), now by choosing the gauge [6]

$$A_j = (-1)^j\,\frac{B}{2}\left(x_1 + x_2 + \alpha_j a\right) \ , \tag{4}$$

such that $\alpha_1 = -\alpha_2 = -1/2$ and $\theta_1 = \theta_2 = \pi/2$. In general, a such procedure is reminiscent to the "chiral" gauge mentioned before [7], but explicit steps [6] are necessary for a better understanding. In this context we shall analyze the $\Delta \neq 1$-generalization of Eq. (1) by using this time a Laurent-series representation for the wavefunction [8]. We shall then use this opportunity to derive energy-polynomials exhibiting a reasonable dependence on a continuous extrapolation of the commensurability parameter like $\hbar^* \in [0, 2\pi]$. It is understood that such extrapolations are useful for further studies concerning thermodynamic properties.

2 Preliminaries and notations

The generalization of Eq. (1) for arbitrary values of the Δ-parameter is given by

$$i\left(\frac{1}{z} + \Delta qz\right)\psi\left(qz\right) - i\left(\frac{z}{q} + \frac{\Delta}{z}\right)\psi\left(q^{-1}z\right) = E\psi\left(z\right) \ . \tag{5}$$

This is produced by Eqs. (A2)-(A8) in Ref. [6], now by choosing an arbitrary Δ-parameter instead of $\Delta = 1$. Our main task is to derive a Q-degree energy-dependent polynomial, say $\widetilde{P}^{(Q)}\left(E; q, \Delta\right)$, which should generate the middle band energies in terms of the roots of the algebraic-equation

$$\widetilde{P}^{(Q)}\left(E; q, \Delta\right) = 0 \ , \tag{6}$$

such that $q^{2Q} = 1$. Of course, for a fixed Q-parameter, one has a certain number, say $N_s(Q)$, of coprime realizations P_s of the P-parameter. Some few examples are $N_s(3) = N_s(4) = N_s(6) = 1$, but $N_s(5) = N_s(8) = N_s(10) = 2$ etc. It is also understood that inserting selected $\hbar^*$-values such as $\hbar^* = 2\pi P_s/Q$ into $\widetilde{P}^{(Q)}\left(E; q, \Delta\right)$ yields a number of $N_s(Q)$ distinct polynomial realizations like $\widetilde{P}_k^{(Q)}\left(E; \Delta\right)$, where $k = 1, 2, \ldots, N_s(Q)$. The energy bands are then produced by the inequalities [9]

$$-2 - 2\Delta^Q \leqslant \widetilde{P}^{(Q)}\left(E; q, \Delta\right) \leqslant 2 + 2\Delta^Q \ , \tag{7}$$

in which the equality-signs are responsible for the band-edges. The $\widetilde{P}_k^{(Q)}$-polynomials can be established by resorting to the transfer matrix approach

[10, 11, 12], to the method of the secular-equation [9, 13], or to the Bethe-ansatz approach [1]. Choosing $\Delta = 1$, proofs have been given that the Q roots of $\widetilde{P}_k^{(Q)}(E;1) = 0$ fulfil the energy-reflection symmetry [14]. This property is a consequence of the $sl_q(2)$-symmetry, which means that $-E$ is an energy-root if E does it. Moreover, such polynomials fulfil the Bender-Dunne symmetry [15], too. This amounts to consider that the wavefunction itself is the generating function of $\widetilde{P}_k^{(Q)}(E;1)$-polynomials referred to above. Furthermore, energy $\widetilde{P}_k^{(Q)}(E;\Delta)$-polynomials for which Δ is an arbitrary positive parameter have also been considered [9, 16]. In addition, explicit results concerning such polynomials have been written down recently for $Q = 1-8$ [17] by using the transfer matrix approach.

3 Applying Laurent series

Now we are ready to derive the continuous extrapolation of the energy-polynomial i.e. $\widetilde{P}^{(Q)}(E;q,\Delta)$, by using Laurent-series. Limiting realizations of such polynomials have been discussed before for $\Delta = 1$ [6], but explicit results concerning "discretized" $\widetilde{P}_k^{(Q)}(E;\Delta)$-polynomials are also available [17], as mentioned before. It is clear that $\widetilde{P}^{(Q)}(E;q,\Delta)$-polynomials have to be established so as to reproduce both $\widetilde{P}^{(Q)}(E;q,1)$- and $\widetilde{P}_k^{(Q)}(E;\Delta)$-limits. On the other hand continuous extrapolations can also be derived by resorting to other methods referred to previously, namely to the secular equation or to the transfer matrix. However, we have to realize, excepting selected $\hbar_s^*$-points, that such extrapolations are not at all identical. Upon further clarification, we shall then proceed by deriving the Laurent-series alternative to the continuous extrapolation, which represents by itself a nontrivial result.

Inserting the Laurent-series

$$\psi(z) = \sum_{n=-\infty}^{+\infty} c_n z^n \ , \tag{7}$$

into Eq. (5) then gives the three-term recurrence relation

$$Ec_n = ic_{n+1}\frac{q^{2n+2}-\Delta}{q^{n+1}} + ic_{n-1}\frac{\Delta q^{2n}-1}{q^n} \ . \tag{8}$$

Of course, integrating Eq. (5) along a closed contour in the complex plane centered at $z = 0$ yields the condition

$$i\left(1-\Delta\right)c_0 + \frac{i}{q}c_{-2}\left(\Delta - q^2\right) = Ec_{-1} \ , \tag{9}$$

by virtue of Eq. (7), which reproduces precisely the $n = -1$ form of Eq. (8). In general, we can then proceed choosing $c_0 = 0$ or $c_{-1} = 0$, thereby preserving the correct $\Delta = 1$-limit. However, this choice works in terms of real q-values, so that it will be hereafter ignored.

Proceeding via

$$c_{-1} = 0 \ , \tag{10}$$

we shall begin by considering that $n \geqslant 0$, in which case

$$c_0 = \frac{1}{1-\Delta} \ . \tag{11}$$

One would then obtain

$$c_n = (-i)^n \prod_{j=0}^{n} \frac{q^j}{q^{2j}-\Delta} R^{(n)} \ , \tag{12}$$

where $R^{(n)}(E;q,\Delta)$ is a polynomial of degree "n" in E satisfying the three-term recurrence relation

$$R^{(n)} = ER^{(n-1)} + R^{(n-2)}(\Delta\Gamma_{2n-2} - 1 - \Delta^2) \ , \tag{13}$$

where

$$\Gamma_n = q^n + \frac{1}{q^n} \ . \tag{14}$$

Now $n \geqslant 1$, such that $R^{(-1)} = 0$ and $R^{(0)} = 1$.

On the other hand one has

$$c_{-n_0} \equiv \widetilde{c}_{n_0} = (-i)^{n_0} (-1)^{n_0 - 1} \prod_{j=1}^{n_0 - 1} \frac{q^j}{q^{2j}-\Delta} L^{(n_0-2)} \ , \tag{15}$$

for $n \leqslant -2$, where $n_0 = |n| \geqslant 2$. This time $L^{(n_0-2)} \equiv L^{(n_0-2)}(E;q,\Delta)$ is an energy polynomial of degree "$n_0 - 2$" obeying the recurrence relation

$$L^{(n_0)} = EL^{(n_0-1)} + L^{(n_0-2)}\left[\Delta\Gamma_{2n_0} - 1 - \Delta^2\right] \ , \tag{16}$$

such that $L^{(-1)} = 0$ and $L^{(0)} = 1$.

After having been arrived at this stage we have to look for an eigenvalue condition enabling us to establish the energy polynomial in terms of Eq. (6), such that $q^{2Q} = 1$. A such condition is given by

$$\widetilde{c}_Q = \frac{\exp i\pi\,(Q+1)}{q^Q} c_Q \ , \tag{17}$$

which results by virtue of a more careful analysis of recurrence relations. This means in turn that the energy polynomial one looks for is given by

$$\widetilde{P}^{(Q)}(E;q,\Delta) = R^{(Q)}(E;q,\Delta) - (1-\Delta)^2 L^{(Q-2)}(E;q,\Delta) \ , \tag{18}$$

which shows that $\widetilde{P}^{(Q)}$ can be established in a well defined manner in terms of $R^{(Q)}$ and $L^{(Q-2)}$. Some few explicit examples are

$$\widetilde{P}^{(1)}(E;q,\Delta) = E \ , \tag{19}$$

$$\widetilde{P}^{(2)}(E;q,\Delta) = E^2 - 2\left(\Delta^2+1\right) + \Delta\left(2+\Gamma_2\right) \ , \tag{20}$$

and

$$\widetilde{P}^{(3)}(E;q,\Delta) = E\left[E^2 - 3\left(\Delta^2+1\right) + \Delta\left(\Gamma_2+\Gamma_4+2\right)\right] \ . \tag{21}$$

The underlying $R^{(n)}$-and $L^{(n)}$-polynomials are given by

$$R^{(1)} = E \ , \tag{22}$$

$$R^{(2)} = E^2 - \Delta^2 - 1 + \Delta\Gamma_2 \ , \tag{23}$$

$$R^{(3)} = E\left[E^2 - 2\left(\Delta^2+1\right) + \Delta\left(\Gamma_2+\Gamma_4\right)\right] \ , \tag{24}$$

and

$$L^{(1)} = E \ , \tag{25}$$

$$L^{(2)} = E^2 + \Delta\Gamma_4 - 1 - \Delta^2 \ , \tag{26}$$

$$L^{(3)} = E\left[E^2 - 2\left(\Delta^2+1\right) + \Delta\left(\Gamma_4+\Gamma_6\right)\right] \ , \tag{27}$$

respectively. Other cases can be treated in a similar manner.

One realizes that energy reflection symmetry is preserved if $\Delta \neq 1$, too. In contradistinction, the Bender-Dunne symmetry ceases to be valid, but Eq. (17) can be viewed as a generalized version of this one.

4 Conclusions

In this paper we succeeded to establish a tractable Laurent-series version of the $\widetilde{P}^{(Q)}(E;q,\Delta)$-polynomial. This polynomial has the meaning of a well defined generalization, which also means that previous results are able to be reproduced as a limiting cases. Such results are able to be combined with recent $\Delta \neq 1$-generalizations concerning the density of states [17]. Putting together results just mentioned above opens the way for updated studies of thermodynamic properties, but transport properties can also be accounted for. In the latter case correlations between spectral and transport properties have to be invoked, too.

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