MONOTONICITY METHODS IN THE NONLINEAR KINETIC THEORY

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Abstract

This paper reviews results on the existence theory of solutions for a class of kinetic models, recently introduced as generalizations of the classical Boltzmann equation. The problem of the existence, uniqueness and positivity of global solutions can be investigated by extending monotonicity methods, developed for solving the classical Boltzmann equation.

1 Introduction

The last years have been marked by an increased interest in the mathematical properties of the non-linear kinetic models, appearing as generalizations of the classical Boltzmann equation [1]. This can be explained by the various applications not only in physics, astrophysics and chemistry (e.g. studies of simple and complex/reacting fluids, granular media, coagulation-fragmentation, formation of planetary rings , galaxy collision) but also in modeling evolution processes in immunology, traffic flow, communication networks, etc.

The above generalized Boltzmann equations are phenomenological or microscopic models that describe the evolution of *populations* (macroscopic systems) of many well individualized, *objects* (e.g. rarefied gas particles, cells networks signals etc.) interacting among themselves. Such interactions are (localized) *microscopic* processes in the following sense: any interaction has a very short duration, with respect to the time-scale of the macroscopic evolution; b) the number of partners of any interaction is very small, with respect to the total number of the components of the population. Depending on the model, an interaction may change the state, nature and/or the number of the participants in interaction. This may result in modifications of the values of the physical quantities characterizing the states of the interacting objects. However, such modifications must be consistent with certain *balance* laws (e.g. conservation /dissipation laws) imposed by the peculiarities of the microscopic processes.

The problem of the existence and uniqueness of solutions to the generalized Boltzmann equations is not only of an academic interest. Indeed, good criteria for the existence of general solutions and a detailed study of the properties of the solutions can be particularly useful in obtaining effective convergent numerical schemes for the models.

The generalized Boltzmann equations have mathematical properties similar to those of the classical Boltzmann equation. Due to this fact, the problem of the existence, uniqueness and positivity of global solutions can be solved by extending, non-trivially, monotonicity methods developed within the framework of the mathematical kinetic theory of the classical Boltzmann equation.

To the best of the author's knowledge, monotonicity methods were introduced in the study of nonlinear kinetic equations by Arkeryd [2] who investigated the space homogeneous Boltzmann equation. Wiesen [3] extended the results of [2] to a space-homogeneous Boltzmann model for a single component real gas with inelastic binary collisions. Recent applications of Arkeryd's method to various models (fluids with stochastic collision laws granular media, reactive gases, fluids with dissipative collisions) were obtained, in particular, by Lachowicz and Pulvirenti [4], Grünfeld [5], De Angelis and Grünfeld [6], [7].

In this paper, we present two such recent results, representative for the application to conservative and dissipative models, aiming to emphasize their common features.

Besides this Introduction, the paper includes three more section. In the next section, Section 2, we present the common features of the generalized Boltzmann model in a rather formal, abstract frame and provide the main examples. Section 3 contains the main existence results and the general scheme of the argument. Section 4 is devoted to the conclusions and open problems.

2 Generalized Boltzmann models

The analysis of various Boltzmann models reveals that, in general, they are described by nonlinear evolution equations (in some ordered Banach space X) of the form

$$\frac{df}{dt} = Df + Q(t, f), \quad t > 0, \tag{1}$$

where the unknown $f = f(t)$ characterizes the state of the macroscopic system at time t. The two terms of the r.h.s. of Eq.(1), Df and $Q(t, f)$ describe the free motion and the contribution of the interaction processes, respectively. From a mathematical point of view, D is the generator of an evolution linear group in X, while $Q(t, \cdot)$ - the so-called Boltzmann operator -, is a nonlinear integral operator in X.

In many situations, we can write $Q(t, \cdot) = Q^+(t, \cdot) - Q^-(t, \cdot)$, where $Q^+(t, \cdot)$ and $Q^-(t, \cdot)$ are *positive* (i.e. $g \geq 0 \Rightarrow Q^\pm(t, g) \geq 0$) and *monotone* (i.e. $g \geq h \geq 0 \Rightarrow Q^\pm(t, g) \geq Q^\pm(t, h) \geq 0$). Moreover, $Q^+(t, \cdot)$ and $Q^-(t, \cdot)$ satisfy certain relations -macroscopic balance laws- determined by the microscopic balance properties.

An important problem related to Eq.(1) is the initial value (i.v.p.) problem. This can take various formulations, depending on the model.

As a first example, we consider the generalized Boltzmann model introduced in [6] for a reacting gas mixture of N species A_i and mass m_i, $1 \leq i \leq N$, with binary reactions (without interaction with photon fields)

$$A_i + A_j \rightarrow A_k + A_l\,, \quad 1 \leq i\,, j\,, k\,, l \leq N\,. \tag{2}$$

Here $i = j = k = l$ corresponds to the non-reactive (elastic) processes. According to the model, for each species i, the gas particles have one internal energy state,E_i, $1 \leq i \leq N$ (this condition is not restrictive, since different internal states of a particle can be treated as particles of distinct species). The reactions are consistent with the conservation of mass, momentum and total energy, i.e., $m_i + m_j = m_k + m_l$, $m_i\mathbf{v} + m_j\mathbf{w} = m_k\mathbf{v}' + m_l\mathbf{w}'$ and $\frac{m_i|\mathbf{v}|^2}{2} + E_i + \frac{m_j|\mathbf{w}|^2}{2} + E_j = \frac{m_k|\mathbf{v}'|^2}{2} + E_k + \frac{m_l|\mathbf{w}'|^2}{2} + E_l$, with $(\mathbf{v}, \mathbf{w})$ the pre-reaction velocities of the particles (i, j) and $(\mathbf{v}', \mathbf{w}')$ the post-reaction velocities of the particles.

In this case, considerations close to the ideas of [8] lead to the following i.v.p.

$$\frac{\partial}{\partial t} f_i = -\mathbf{v}\cdot\nabla_{\mathbf{x}} f_i + G_i(f) - L_i(f)\,,\ \ f_i(0,\mathbf{x},\mathbf{v}) = f_{0,i}(\mathbf{x},\mathbf{v}), \tag{3}$$

where $f_i := f_i(t,\mathbf{x},\mathbf{v})$ is the one-particle distribution function of the particles of species i, and depends on time - t position - $\mathbf{x}$ and velocity - $\mathbf{v}$. In (3),

$$G_i(f)(t,\mathbf{x},\mathbf{v}) := \\ \sum_{j,k,l=1}^{N} \int_{R^3\times R^3\times S^2} p_{kl,ij}(\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n}) f_k(t,\mathbf{x},\mathbf{v}_{kl,ij}) f_l(t,\mathbf{x}+\mathbf{y},\mathbf{w}_{kl,ij}) d\mathbf{y}d\mathbf{w}d\mathbf{n}$$

$$L_i(f)(t,\mathbf{x},\mathbf{v}) := \\ \sum_{j,k,l=1}^{N} \int_{R^3\times\ R^3\times S^2} r_{kl,ij}(\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n}) f_i(t,\mathbf{x},\mathbf{v}) f_j(t,\mathbf{x}+\mathbf{y},\mathbf{w}) d\mathbf{y}d\mathbf{w}d\mathbf{n}\,.$$

Here, $S^2 = \{\mathbf{n}\in\ R^3 : |\mathbf{n}| = 1\}$.Also, denoting $t_{kl,ij} := \frac{m_i m_j}{2(m_i+m_j)} |\mathbf{v}-\mathbf{w}|^2 + E_i + E_j - E_k - E_l$, the maps $\mathbf{v}_{kl,ij}\ \mathbf{w}_{kl,ij} :\ R^3 \times\ R^3 \times S^2 \to\ R^3$ are defined by

$$\mathbf{v}_{kl,ij}(\mathbf{v},\mathbf{w},\mathbf{n}) := \frac{m_i\mathbf{v}+m_j\mathbf{w}}{m_i+m_j} + \frac{2^{1/2}(m_l)^{1/2}}{m_k^{1/2}(m_i+m_j)^{1/2}} t_{kl,ij}(\mathbf{v},\mathbf{w})^{1/2}\mathbf{n}$$

$$\mathbf{w}_{kl,ij}(\mathbf{v},\mathbf{w},\mathbf{n}) := \frac{m_i\mathbf{v}+m_j\mathbf{w}}{m_i+m_j} - \frac{2^{1/2}(m_k)^{1/2}}{m_l^{1/2}(m_i+m_j)^{1/2}} t_{kl,ij}(\mathbf{v},\mathbf{w})^{1/2}\mathbf{n}$$

for $(\mathbf{v},\mathbf{w}) \in \mathcal{D}_{ij,kl} := \{(\mathbf{v},\mathbf{w}) \in R^3\times R^3 : t_{kl,ij} \geq 0\}$ and $\mathbf{v}_{kl,ij}(\mathbf{v},\mathbf{w},\mathbf{n}) = \mathbf{w}_{kl,ij}(\mathbf{v},\mathbf{w},\mathbf{n}) = 0$, otherwise. Here $\mathbf{v}_{kl,ij}$ and $\mathbf{w}_{kl,ij}$ define velocities of the particles emerging from reaction (2) as functions of the velocities $\mathbf{v}$ and $\mathbf{w}$, of the particles entering in the reaction. Moreover, $p_{kl,ij}, r_{kl,ij} : R^3\times R^3\times R^3\times S^2 \to [0,\infty)$, are given measurable functions (the so-called reaction laws defined by the microscopic interaction processes). By definition, $p_{kl,ij}(\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n}) = r_{kl,ij}(\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n}) = 0$ for $(\mathbf{v},\mathbf{w}) \notin \mathcal{D}_{ij,kl}$. Furthermore, due to the properties of the microscopic interactions,

$$p_{kl,ij}(\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n}) = p_{kl,ji}(\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n}) = p_{lk,ij}(\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n})$$

$$r_{kl,ij}(\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n}) = r_{kl,ji}(\mathbf{y},\mathbf{w},\mathbf{v},\mathbf{n}) = r_{lk,ij}(\mathbf{y},\mathbf{v},\mathbf{w},-\mathbf{n}),$$

$$p_{kl,ij}(\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n}) = p_{kl,ij}(-\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n}), \quad r_{kl,ij}(\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n}) = r_{kl,ij}(-\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n}).$$

In addition,

$$\int_{R^3\times R^3\times S^2} \varphi(\mathbf{v},\mathbf{w}) p_{kl,ij}(\mathbf{v},\mathbf{w},\mathbf{n})\psi(\mathbf{v}_{kl,ij},\mathbf{w}_{kl,ij}) d\mathbf{v}d\mathbf{w}d\mathbf{n}$$

$$= \int_{R^3\times R^3\times S^2} \varphi(\mathbf{v}_{ij,kl},\mathbf{w}_{ij,kl}) r_{ij,kl}(\mathbf{v},\mathbf{w},\mathbf{n})\psi(\mathbf{v},\mathbf{w}) d\mathbf{v}d\mathbf{w}d\mathbf{n}$$

for $(\psi,\varphi) : R^3 \times R^3 \to [0,\infty)$, measurable.

Formally, the above properties imply the so-called bulk conservation of mass, momentum and internal energy, expressed by

$$\sum_{i=1}^{N} \int_{R^3\times R^3} \Psi_i^{(j)}(\mathbf{x},\mathbf{v}) f_i(t,\mathbf{x},\mathbf{v}) d\mathbf{x}d\mathbf{v} = \sum_{i=1}^{N} \int_{R^3\times R^3} \Psi_i^{(j)}(\mathbf{x},\mathbf{v}) f_i(0,\mathbf{x},\mathbf{v}) d\mathbf{x}d\mathbf{v}, \tag{4}$$

where $\Psi_i^{(0)}(\mathbf{x},\mathbf{v}) := m_i$, $\Psi_i^{(4)}(\mathbf{x},\mathbf{v}) := m_i |\mathbf{v}|^2/2 + E_i$, $\Psi_i^{(j)}(\mathbf{x},\mathbf{v}) := m_i v_j$, $j = 1,2,3$, with v_j the components of $\mathbf{v}$.

Due to the physical meaning of the quantities f_i, it is natural to formulate the Cauchy problem for Ec. (3) in the space $X_a := (L^1(R^3 \times R^3; d\mathbf{x}d\mathbf{v}))^N$ (with L^1 -real), equipped with the norm $\|f\| = \sum_{i=1}^N m_i \|f_i\|_{L^1}$ and the order $\leq$ induced by the natural order of L^1-real.

Defining, $f := (f_1,...,f_N)$, and $D := (D_1,....,D_N)$ with $(Df)_i := -\mathbf{v}\cdot\nabla_{\mathbf{x}} f_i$, and setting $Q^+ : (G_1,...,G_N)$ and $Q^- := (L_1,...,L_N)$, one can see that Ec. (3) can be written in the form (1), where the positive and negative parts of the Boltzmann operators are monotone and (4) provides the additional macroscopic balance laws.

Our second example refers to a one-component gas of classical particles undergoing binary dissipative collisions. According to the model, the mass and momentum are conserved by collisions, but the energy is dissipated [7]. Specifically, for this model, the post-collision velocities $\mathbf{v}^*$ and $\mathbf{w}^*$ are defined in terms of pre-collision velocities $\mathbf{v}$, $\mathbf{w}$ by $\mathbf{v}^* = \mathbf{v} + (1-\beta)\langle\mathbf{w}-\mathbf{v},\mathbf{n}\rangle\mathbf{n}$ and $\mathbf{w}^* = \mathbf{w} - (1-\beta)\langle\mathbf{w}-\mathbf{v},\mathbf{n}\rangle\mathbf{n}$,where $\beta \in [0,1/2)$ is the parameter characterizing the energy dissipation. In this case, following Boltzmann's scheme and ideas from [9], [10] we obtain the i.v.p

$$\frac{\partial}{\partial t} F = -\mathbf{v}\cdot\nabla_{\mathbf{x}} F + G_d(F) - L_d(F), \quad F(0,\mathbf{x},\mathbf{v}) = F_0(\mathbf{x},\mathbf{v}) \geq 0. \tag{5}$$

The unknown $F = F(t, \mathbf{x}, \mathbf{v})$ is the one-particle distribution function (at moment t, position $\mathbf{x}$, velocity $\mathbf{v}$) of the gas particles. Here

$$G_d(F)(t, \mathbf{x}, \mathbf{v}) := \frac{1}{(1-2\beta)^{1+\gamma}} \int_0^R \int_{R^3 \times S^2} |\langle \mathbf{n}, \mathbf{w} - \mathbf{v} \rangle|^\gamma \times$$

$$\times P(r, \mathbf{n}) F(t, \mathbf{x}, \mathbf{w}^+) F(t, \mathbf{x} + r\mathbf{n}, \mathbf{w}^-) d\mathbf{n}\, d\mathbf{w}\, dr \tag{6}$$

$$L_d(F)(t, \mathbf{x}, \mathbf{v}) := F(t, \mathbf{x}, \mathbf{v}) \int_0^R \int_{R^3 \times S^2} |\langle \mathbf{n}, \mathbf{w} - \mathbf{v} \rangle|^\gamma \times$$

$$\times P(r, \mathbf{n}) F(t, \mathbf{x} + r\mathbf{n}, \mathbf{w}) d\mathbf{n}\, d\mathbf{w}\, dr \tag{7}$$

where $<,>$ denotes the scalar product on R^3 and $P: \; R_+ \times S^2 \to \; R_+$ is a given, measurable function with property , $P(r, \mathbf{n}) = P(r, -\mathbf{n})$; $0 \leq \gamma \leq 1$, $R > 0$. Further, $\mathbf{w}^\pm := \mathbf{w} \pm \left(\frac{1-\beta}{1-2\beta}\right) \langle \mathbf{w} - \mathbf{v}, \mathbf{n} \rangle \mathbf{n}, \;\; \mathbf{n} \in S^2$.

Let $M(f)(t) = \int_{R^3 \times S^3} F(t, \mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}$, $\mathbf{P}(f)(t) = \int_{R^3 \times R^3} \mathbf{v}\, F(t, \mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}$ and $E(f)(t) = \int_{R^3 \times R^3} \frac{1}{2} |\mathbf{v}|^2 F(t, \mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}$ denote the bulk mass, momentum and energy, respectively. It can be checked that, at least formally, one has the conservation of mass and momentum, respectively $M(f_0) = M(f)(t)$, $\mathbf{P}(f_0) = \mathbf{P}(f)(t)$, and the dissipation relation

$$E(f_0) = E(f)(t) + R(t, f), \tag{8}$$

where

$$0 \leq R(t; f) == \frac{1}{2}\beta(1-\beta) \int_0^t ds \int_0^R \int_{R^3 \times \;\; R^3 \times \;\; R^3 \times S^2} |\langle \mathbf{n}, \mathbf{w} - \mathbf{v} \rangle|^{\gamma+2}$$

$$\times P(r, \mathbf{n}) F(s, \mathbf{x}, \mathbf{v}) F(s, \mathbf{x} + r\mathbf{n}, \mathbf{w})\, d\mathbf{n}\, d\mathbf{w}\, d\mathbf{v}\, d\mathbf{x}\, dr\, . \tag{9}$$

Observe that if $\beta = 0$ and $P(r, \mathbf{n}) = \delta(r)$, then Ec. (5) reduces to the classical Boltzmann equation.

Formulating (5) in $X = L^1 = L^1(\; R^3 \times \;\; R^3; \; d\mathbf{x}\, d\mathbf{v})$-real, and setting $D := -\mathbf{v} \cdot \nabla_{\mathbf{x}}$, $Q^+ := G_d$, $Q^- := L_d$, one can see that the obtained problem is also of the form (1).

In the following section, we examine the existence of general solutions for problems (3) and (5)

3 Existence theory

The monotonicity properties of the positive and negative parts of the Boltzmann operator enables the use of the monotone approximation theory to solve the general Boltzmann equation. To this end, one applies a generalized version of the Arkeryd's compensated scheme [2]. In the case of the examples presented in Section 2, the results are based on rather general assumptions.

Suppose, in the case of (3), that there exist constants $c_q > 0$ and $0 \leq q \leq 1$, such that

$$\int_{S^2} r_{kl,ij}(\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{n}) d\mathbf{n} \leq c_q \left[1 + |\mathbf{v}|^2 + |\mathbf{w}|^2\right]^q .$$

Theorem 1
Let $(1+|\mathbf{v}|^2)^2 f_0 \in X_a$. *Then for each* $T > 0$, *Problem (3) has a unique mild solution* $0 \leq f(t) \in X_a$ *on* $[0,T]$ *satisfying*

$$0 \leq (1 + |\mathbf{v}|^2)^2 f(t) \in X_a,$$

and the conservation relations (4). Moreover, there is some constant b_T *depending on* f_0 *and* T *such that*

$$\left\|(1 + |\mathbf{v}|^2)^2 f(t)\right\| \leq b_T \left\|(1 + |\mathbf{v}|^2)^2 f_0\right\|, \quad 0 \leq t \leq T.$$

Theorem 2. *Suppose there is constant* $c_0 > 0$, *such that* $P(r,\mathbf{n}) \leq c_0 r^2$, $r \geq 0$, $\mathbf{n} \in S^2$. *If,*

$$0 \leq (1 + |\mathbf{v}|^2)^3 F_0 \in L^1 := L^1(R^3 \times R^3; d\mathbf{x}d\mathbf{v}),$$

then problem (5) has a unique mild solution F *in* $[0,T]$, *with*

$$0 \leq (1 + |\mathbf{v}|^2)^3 F(t) \in L^1,$$

and F *verifies the balance laws (8).*

Remark: in the particular case $\beta = 0$, Theorem 2 states the existence, uniqueness and positivity of solutions for the so-called Boltzmann equation with "averaged collision laws" [8].

The proofs are rather involved and based on many technical estimations. However they follow the same scheme. In the remaining of this section we give a brief (formal) description of the main steps of this scheme. For detailed proofs of the above results the reader is referred to [6] and [7]. Our considerations will refer to the abstract model (1):

Step 1. One eliminates the linear part in (1). To this end, a simple integration on characteristics reduces problem (1) with $D \neq 0$ to the simpler one with $D = 0$

$$\frac{df}{dt} = Q^+(t, f) - Q^-(t, f), \qquad f(0) = f_0. \tag{10}$$

Step 2. Considering (for simplification) that Eq. (1) is supplemented with balance laws of the form (8), one can write the balance laws as

$$E(f_0)A = E(f)(t)A + R(t, f)A \tag{11}$$

with $(-A)$ the generator of a suitable positive, monotone, C_o semi-group, Then if $\lambda > 0$ is sufficiently large, Eq. (10) is equivalent with the following equation with *positive and monotone* terms

$$\frac{df}{dt} + \lambda E(f_0)Af = Q^+(t, f) + \{\lambda[E(f)(t)f + R(t, f)]Af - Q^-(t, f)\}, \tag{12}$$

supplemented with condition (11).

Further, one uses the fact that X is monotone complete [11], i.e. if the sequence $\{g_n\}_{n\in N}$ is positive, monotone (increasing) and bounded in X, then it is convergent in X.

Step 3. One introduces convenient regular monotone approximations $Q_n^{\pm}(t, \cdot) \nearrow Q^{\pm}(t, \cdot)$, $R_n(t, f) \nearrow R(t, f)$ as $n \to \infty$ (by introducing suitable "truncated" kernels in the original Boltzmann operators).

Step 4 One considers the following approximations of problem (10)

$$\frac{df}{dt} + \lambda E(f_0)Af = Q_n^+(t, f) + \{\lambda[E(f)(t) + R_n(t, f)]Af - Q_n^-(t, f)\}, \; f(0) = f_0, \tag{13}$$

$$E(f_0)A = E(f)(t)A + R_n(t, f)A \tag{14}$$

and

$$\frac{df}{dt} + \lambda E(f_0)Af = Q_n^+(t, f) + \{\lambda[E(f)(t) + R_n(t, f)]Af - Q^-(t, f)\}, \; f(0) = f_0, \tag{15}$$

supplemented with condition (14).

Step 5. Due to the choice of $Q_n^{\pm}$, one can apply the Banach fixed point theorem to obtain a unique positive solution F_n of Ec.(13). Now, F_n seems to be an approximate solution of (10). However we do not know whether F_n converges to some solution of Ec.(12) , as $n \rightharpoonup \infty$.

Step 6. To refine the above approximation, one uses the monotonicity properties of the terms in the r.h.s of Eq. (15) and (13), and for each n fixed, one constructs suitable iterations $\{f_{n,i}(t)\}_{i\in N}$ of Eq. (15) such that $f_{n,0}(t) \leq f_{n,1}... \leq f_{n,i} \leq$ ...and $f_{n,i}(t) \leq F_n(t)$. Then by the monotone completeness of X, one finds that there is $f_n(t) = \lim_{i\to\infty} f_{n,i}(t) \leq F_n(t)$.

Step. 7 Due to the balance laws, $\{F_n(t)\}_{n\in N}$ is bounded. One finds that $\{f_n(t)\}_{n\in N}$ is also monotone and bounded, hence again by monotone completeness there is $\hat{f}(t) = \lim_{n\to\infty} f_n(t)$.

Step 8. One proves that $Q_n^-(t, f_n) \nearrow Q^-(t, \hat{f})$, $R_n(t, f_n) \nearrow R(t, \hat{f})$, $E(f_n)(t) \nearrow E(\hat{f})(t)$ and

$$E(f_0) \geq E(\hat{f})(t) + R(t, \hat{f}). \tag{16}$$

Thus $\hat{f}$ is solution of Eq. (12) if we have equality in (16). One proves that (16) is satisfied with the equality sign by applying technical estimations on "moments", (this imposes the boundedness of the higher order moments of the initial data).

Step 9 The uniqueness of the solution results by construction, since $\hat{f}$ is "smaller" than any other positive solution of Ec.(12).

4 Concluding remarks

In this paper, we presented two examples of existence results for generalized Boltzmann models obtained by monotonicity methods. The results can be completed with other applications in the domain of generalized Boltzmann models. Moreover, these methods seem equally useful to investigate evolution problems from other fields of science.

On the other hand the results presented in Section 2 describe only partially the properties of the models considered. They must be completed by a thorough study of other properties of the models, e.g. the existence of stationary or/and equilibrium solutions, Lyapunov functionals, H-theorems (see e.g. [7]), asymptotic properties, construction of effective numerical methods.

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