Uncertainty relations in the theory of open quantum systems

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Abstract

In the framework of the Lindblad theory for open quantum systems we derive analytical expressions of the Heisenberg and Schrödinger generalized uncertainty functions for a particle moving in a harmonic oscillator potential. The particle is initially in an arbitrary correlated coherent state and interacts with an environment at finite temperature. We analyze the relative importance of quantum and thermal fluctuations and show that the system evolves from a quantum-dominated to a thermal-dominated state in a time which is of the same order as the decoherence time.

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1 Introduction

There is a large amount of papers concerned with the question of modification and generalization of the uncertainty principle, in particular on modified uncertainty relations which incorporate the effect of environmentally induced fluctuations [1, 2, 3, 4]. These authors used quantum Brownian models consisting of a particle moving in a potential and linearly coupled to a bath of harmonic oscillators in a thermal state.

In the present work we study, in the framework of the Lindblad theory for open quantum systems, the Heisenberg and Schrödinger generalized uncertainty functions and the role of quantum and thermal fluctuations during the evolution of a particle moving in a harmonic oscillator potential and interacting with an environment. In Sec. 2 we remind the basic results concerning the evolution of the damped harmonic oscillator in the Lindblad theory. Then in Sec. 3 we derive analytical expressions for the finite temperature Heisenberg and Schrödinger generalized uncertainty functions [5]. We consider the case of a thermal bath and take the correlated coherent and squeezed states, in particular the coherent states, as initial states. In Sec. 4 we discuss the relative importance of quantum and thermal fluctuations in the evolution of the system towards equilibrium with the aim of clarifying the meaning of quantum, classical and thermal regimes, motivated by the necessity of understanding the process of decoherence via interaction with the environment and the general problem of the transition from quantum to classical behaviour [5]. In Sec. 5 we discuss our results and a summary and concluding remarks are given in Sec. 6.

2 Lindblad master equation for damped harmonic oscillator

It is generally thought that quantum dynamical semigroups are the basic tools to introduce dissipation in quantum mechanics [6, 7]. In Markovian approximation and for weakly damped systems, the most general form of the generators of such semigroups was given by Lindblad [8]. This formalism has been studied extensively for the case of damped harmonic oscillators [7, 9, 10, 11, 12] and a phase space representation for the open quantum systems was given in Refs. [13, 14]. In Lindblad axiomatic formalism the irreversible time evolution of the open system is described by the following general quantum Markovian master equation for the density operator $\rho(t)$ [8]:

$$\frac{d\rho(t)}{dt}=-\frac{i}{\hbar}[H,\rho(t)]+\frac{1}{2\hbar}\sum_j([V_j\rho(t),V_j^\dagger]+[V_j,\rho(t)V_j^\dagger]). \tag{1}$$

Here H is the Hamiltonian operator of the system and V_j, $V_j^\dagger$ are operators on the Hilbert space of the Hamiltonian, which model the environment. In the case of an exactly solvable model for the damped harmonic oscillator, the two possible operators V_1 and V_2 are taken as linear polynomials in coordinate q and momentum p [7, 9] and the harmonic oscillator Hamiltonian H is chosen of the general quadratic form

$$H=H_0+\frac{\mu}{2}(qp+pq),\quad H_0=\frac{1}{2m}p^2+\frac{m\omega^2}{2}q^2. \tag{2}$$

With these choices the master equation (1) takes the following form [7, 9]:

$$\begin{aligned}\frac{d\rho}{dt}&=-\frac{i}{\hbar}[H_0,\rho]-\frac{i}{2\hbar}(\lambda+\mu)[q,\rho p+p\rho]+\frac{i}{2\hbar}(\lambda-\mu)[p,\rho q+q\rho]\\&-\frac{D_{pp}}{\hbar^2}[q,[q,\rho]]-\frac{D_{qq}}{\hbar^2}[p,[p,\rho]]+\frac{D_{pq}}{\hbar^2}([q,[p,\rho]]+[p,[q,\rho]]).\end{aligned} \tag{3}$$

The quantum diffusion coefficients D_{pp}, D_{qq}, D_{pq} and the dissipation constant λ satisfy the following fundamental constraints [7, 9]: $D_{pp}>0, D_{qq}>0$ and

$$D_{pp}D_{qq}-D_{pq}^2\geq\frac{\lambda^2\hbar^2}{4}. \tag{4}$$

In the particular case when the asymptotic state is a Gibbs state $\rho_G(\infty)=e^{-\frac{H}{kT}}/\mathrm{Tr}e^{-\frac{H}{kT}}$, these coefficients can be written as [7, 9]

$$D_{pp}=\frac{\lambda+\mu}{2}\hbar m\omega\coth\frac{\hbar\omega}{2kT},\quad D_{qq}=\frac{\lambda-\mu}{2}\frac{\hbar}{m\omega}\coth\frac{\hbar\omega}{2kT},\quad D_{pq}=0, \tag{5}$$

where T is the temperature of the thermal bath. In this case, the fundamental constraints are satisfied only if $\lambda>\mu$ and

$$(\lambda^2-\mu^2)\coth^2\frac{\hbar\omega}{2kT}\geq\lambda^2. \tag{6}$$

We notice from this relation that if $T=0$, then we must take $\mu=0$.

By using the complete positivity property of the Lindblad model, in Refs. [7, 9] the following inequality was obtained for all values of $t\geq0$:

$$D_{pp}\sigma_{qq}(t)+D_{qq}\sigma_{pp}(t)-2D_{pq}\sigma_{pq}(t)\geq\frac{\hbar^2\lambda}{2}. \tag{7}$$

The inequality (7) represents a restriction connecting the values of the variances $\sigma_{qq}(t)$ and $\sigma_{pp}(t)$ and covariance $\sigma_{pq}(t)$ of coordinate and momentum with the environment coefficients. In the case of a thermal bath, when the environment coefficients have the form (5), the condition (7) becomes

$$[(\lambda+\mu)m\omega\sigma_{qq}(t)+(\lambda-\mu)\frac{\sigma_{pp}(t)}{m\omega}]\coth\frac{\hbar\omega}{2kT}\geq\hbar\lambda. \tag{8}$$

We have found in Ref. [10] that the inequality (7) is equivalent with the generalized uncertainty relation at any time t

$$\sigma_{qq}(t)\sigma_{pp}(t)-\sigma_{pq}^2(t)\geq\frac{\hbar^2}{4}, \tag{9}$$

if the initial values $\sigma_{qq}(0),\sigma_{pp}(0)$ and $\sigma_{pq}(0)$ satisfy this inequality for $t=0$. The relation (4) is a necessary condition for the generalized uncertainty inequality (9) to be fulfilled.

From the master equation (3) we can obtain the equations of motion for the variances and covariance of coordinate and momentum [7, 9] which are needed to calculate the uncertainty functions. Introducing the notations

$$X(t)=\begin{pmatrix} m\omega\sigma_{qq}(t)\\ \sigma_{pp}(t)/m\omega\\ \sigma_{pq}(t)\end{pmatrix},\quad D=\begin{pmatrix}2m\omega D_{qq}\\ 2D_{pp}/m\omega\\ 2D_{pq}\end{pmatrix}, \tag{10}$$

the solutions of these equations can be written in the form [7, 9]

$$X(t)=(Te^{Kt}T)(X(0)-X(\infty))+X(\infty), \tag{11}$$

where the matrices T and K are given by (we consider the underdamped case $\omega>\mu$ and introduce the notation $\Omega^2=\omega^2-\mu^2$)

$$T=\frac{1}{2i\Omega}\begin{pmatrix}\mu+i\Omega & \mu-i\Omega & 2\omega\\ \mu-i\Omega & \mu+i\Omega & 2\omega\\ -\omega & -\omega & -2\mu\end{pmatrix},\quad K=\begin{pmatrix}-2(\lambda-i\Omega) & 0 & 0\\ 0 & -2(\lambda+i\Omega) & 0\\ 0 & 0 & -2\lambda\end{pmatrix} \tag{12}$$

and $X(\infty)=-(TK^{-1}T)D$. In the case of a thermal bath with coefficients (5), the asymptotic values ($t\to\infty$) of $\sigma_{qq}(t),\sigma_{pp}(t),\sigma_{pq}(t)$ reduce to [7, 9]

$$\sigma_{qq}(\infty)=\frac{\hbar}{2m\omega}\coth\frac{\hbar\omega}{2kT},\quad \sigma_{pp}(\infty)=\frac{\hbar m\omega}{2}\coth\frac{\hbar\omega}{2kT},\quad \sigma_{pq}(\infty)=0. \tag{13}$$

3 Generalized uncertainty functions

We consider a harmonic oscillator with an initial wave function

$$\Psi(x)=(\frac{1}{2\pi\sigma_{qq}(0)})^{\frac14}\exp[-\frac{1}{4\sigma_{qq}(0)}(1-\frac{2i}{\hbar}\sigma_{pq}(0))(x-\sigma_q(0))^2+\frac{i}{\hbar}\sigma_p(0)x], \tag{14}$$

where $\sigma_{qq}(0)$ is the initial spread, $\sigma_{pq}(0)$ the initial covariance, and $\sigma_q(0)$ and $\sigma_p(0)$ are the averaged initial position and momentum of the Gaussian wave packet. As initial state we take a correlated coherent state (squeezed coherent state) which is represented by the Gaussian wave packet (14) in the coordinate representation with the variances and covariance of coordinate and momentum

$$\sigma_{qq}(0)=\frac{\hbar\delta}{2m\omega},\quad \sigma_{pp}(0)=\frac{\hbar m\omega}{2\delta(1-r^2)},\quad \sigma_{pq}(0)=\frac{\hbar r}{2\sqrt{1-r^2}}. \tag{15}$$

Here, δ is the squeezing parameter which measures the spread in the initial Gaussian packet and $r = r(0)$, $|r| < 1$ is the correlation coefficient at time $t = 0$. The correlation coefficient is defined as

$$r(t) = \frac{\sigma_{pq}(t)}{\sqrt{\sigma_{qq}(t)\sigma_{pp}(t)}}. \tag{16}$$

The initial values (15) correspond to a so-called minimum uncertainty state, since they fulfil the generalized uncertainty relation with equal sign

$$\sigma_{qq}(0)\sigma_{pp}(0) - \sigma_{pq}^2(0) = \frac{\hbar^2}{4}. \tag{17}$$

For $\delta = 1$ and $r = 0$ the correlated coherent state becomes a Glauber coherent state. With the initial values (15) the condition (8) for $t = 0$ takes the form ($\epsilon \equiv \hbar\omega/2kT$):

$$[(\lambda+\mu)\delta + (\lambda-\mu)\frac{1}{\delta(1-r^2)}]\coth\epsilon \geq 2\lambda. \tag{18}$$

One can easily show that

$$[(\lambda+\mu)\delta + (\lambda-\mu)\frac{1}{\delta}] \geq 2\sqrt{\lambda^2-\mu^2}. \tag{19}$$

It follows that if relation (6) is satisfied, then relation (18) is also satisfied (since $|r| < 1$) and, therefore, for a given temperature T and for any parameters δ and r the inequality (6) alone determines the range of values of the parameters λ and μ.

3.1 Heisenberg uncertainty function

For simplicity we set $r = 0$ in this Subsection. With the variances given by Eq. (11) we calculate the Heisenberg uncertainty function $U(t) = \sigma_{qq}(t)\sigma_{pp}(t)$ for finite temperature and obtain:

$$\begin{aligned}U(t) = \frac{\hbar^2}{4}\{e^{-4\lambda t}[1-(\delta+\frac{1}{\delta})\coth\epsilon + \coth^2\epsilon + \frac{\omega^2}{4\Omega^4}[\omega^2(\delta-\frac{1}{\delta})^2\sin^2(2\Omega t)\\ +2\mu^2[(\delta-\coth\epsilon)^2+(\frac{1}{\delta}-\coth\epsilon)^2]\cos(2\Omega t)(\cos(2\Omega t)-1)\\ +4\mu^2(\delta-\coth\epsilon)(\frac{1}{\delta}-\coth\epsilon)(1-\cos(2\Omega t))\\ +2\mu\Omega[(\delta-\coth\epsilon)^2-(\frac{1}{\delta}-\coth\epsilon)^2]\sin(2\Omega t)(1-\cos(2\Omega t))]]\\ +e^{-2\lambda t}\coth\epsilon[(\delta+\frac{1}{\delta}-2\coth\epsilon)\frac{\omega^2-\mu^2\cos(2\Omega t)}{\Omega^2}+(\delta-\frac{1}{\delta})\frac{\mu\sin(2\Omega t)}{\Omega}]+\coth^2\epsilon\}.\end{aligned} \tag{20}$$

For $\mu = 0$ and $\delta = 1$ this expression takes the form

$$U(t) = \frac{\hbar^2}{4}\{e^{-2\lambda t} + \coth\epsilon(1-e^{-2\lambda t})\}^2. \tag{21}$$

Here the first term is of quantum nature, whereas the second term is of thermal nature. Their contributions to the uncertainty of the system arise from quantum and thermal fluctuations, respectively.

In the case of $T = 0$ ($\coth\epsilon = 1$) we have to take, according to Eq. (6), $\mu = 0$ and then we obtain from Eq. (20):

$$U_0(t) = \frac{\hbar^2}{4}\{1 + e^{-4\lambda t}[2 - (\delta + \frac{1}{\delta}) + \frac{1}{4}(\delta - \frac{1}{\delta})^2 \sin^2(2\omega t)] + e^{-2\lambda t}[(\delta + \frac{1}{\delta} - 2)\}. \quad (22)$$

We see in this expression that the leading term is given by $\hbar^2/4$ (the Heisenberg contribution) followed, for squeezed states $\delta \neq 1$, by terms describing both decay and oscillatory behaviour, representing quantum fluctuations alone (since $T = 0$). For $\delta = 1$ we obtain in the zero-temperature case $U_0(t) = \hbar^2/4$.

3.2 Schrödinger uncertainty function

With the variances given by Eq. (11) we calculate now the Schrödinger generalized uncertainty function $\sigma(t) = \sigma_{qq}(t)\sigma_{pp}(t) - \sigma_{pq}^2(t)$ and obtain:

$$\sigma(t) = \frac{\hbar^2}{4}\{e^{-4\lambda t}[1 - (\delta + \frac{1}{\delta(1-r^2)})\coth\epsilon + \coth^2\epsilon]$$
$$+e^{-2\lambda t}\coth\epsilon[(\delta + \frac{1}{\delta(1-r^2)} - 2\coth\epsilon)\frac{\omega^2 - \mu^2\cos(2\Omega t)}{\Omega^2}$$
$$+(\delta - \frac{1}{\delta(1-r^2)})\frac{\mu\sin(2\Omega t)}{\Omega} + \frac{2r\mu\omega(1-\cos(2\Omega t))}{\Omega^2\sqrt{1-r^2}}] + \coth^2\epsilon\}. \quad (23)$$

When the initial state is the usual coherent state ($\delta = 1, r = 0$) and $\mu = 0$, we obtain

$$\sigma(t) = \frac{\hbar^2}{4}\{e^{-2\lambda t} + \coth\epsilon(1 - e^{-2\lambda t})\}^2, \quad (24)$$

which is identical with Eq. (21) for $U(t)$.

We consider now the particular case when the temperature of the thermal bath is $T = 0$. Then we have to set also $\mu = 0$ (cf. Eq. (6)) and the uncertainty function $\sigma(t)$ takes the following form from Eq. (23):

$$\sigma_0(t) = \frac{\hbar^2}{4}\{1 + (e^{-4\lambda t} - e^{-2\lambda t})[2 - (\delta + \frac{1}{\delta(1-r^2)})]\}. \quad (25)$$

We see that in this expression the leading term is given by $\hbar^2/4$ (the Heisenberg contribution) followed by terms representing quantum fluctuations alone (since $T = 0$). Compared to Eq. (22), where these terms describe both decay and oscillating behaviour, in Eq. (25) the terms representing the quantum fluctuations describe only a decay behaviour. When the initial state is the usual coherent state ($\delta = 1, r = 0$), the uncertainty function takes again the most simple form $\sigma_0(t) = \hbar^2/4$ for all times.

4 Transition from quantum mechanics to classical statistical mechanics

a) $t = 0$: When the initially uncorrelated condition is assumed valid, we have $\sigma(0) = U(0) = \hbar^2/4$, according to Eq. (17).

b) $t \gg \lambda^{-1}$ (very long times): $\sigma(t)$ and $U(t)$ are insensitive to λ, μ, δ and r and approach

$$\sigma^{BE} = U^{BE} = \frac{\hbar^2}{4} \coth^2 \epsilon, \tag{26}$$

which is a Bose-Einstein relation for a system of bosons in equilibrium at temperature T (quantum statistical mechanics). Again $T = 0$ is the limit of pure quantum fluctuations,

$$\sigma_0 = U_0 = \frac{\hbar^2}{4}, \tag{27}$$

which is the quantum Heisenberg relation and high T ($T \gg \hbar\omega/k$) is the limit of pure thermal fluctuations,

$$\sigma^{MB} = U^{MB} = (\frac{kT}{\omega})^2, \tag{28}$$

which is a Maxwell-Boltzmann distribution for a system approaching a classical limit. The formula (26) interpolates between the two results (27) at $T = 0$ and (28) at $T \gg \hbar\omega/k$.

c) $r = 0$: At short times ($\lambda t \ll 1, \Omega t \ll 1$), we obtain from Eqs. (20) and (23) (where we take $r = 0$ and the correlated coherent initial state becomes a pure squeezed state):

$$\sigma(t) = U(t) = \frac{\hbar^2}{4}\{1 + 2[\lambda(\delta + \frac{1}{\delta}) \coth \epsilon + \mu(\delta - \frac{1}{\delta}) \coth \epsilon - 2\lambda]t\}. \tag{29}$$

The time when thermal fluctuations overtake quantum fluctuations is

$$t_d = \frac{1}{2[\lambda(\delta + \frac{1}{\delta}) \coth \epsilon + \mu(\delta - \frac{1}{\delta}) \coth \epsilon - 2\lambda]}. \tag{30}$$

According to the theory of Halliwell [1, 2] and Hu [3, 4], we expect this time to be equal to the decoherence time scale, which is not yet calculated for the damped harmonic oscillator in the Lindblad model for open quantum systems.

i) At temperature $T = 0$ the uncertainty (29) becomes ($\mu = 0$)

$$\sigma_0(t) = U_0(t) = \frac{\hbar^2}{4}\{1 + 2\lambda(\delta + \frac{1}{\delta} - 2)t\} \tag{31}$$

and

$$t_d = \frac{1}{2\lambda(\delta + \frac{1}{\delta} - 2)}. \tag{32}$$

ii) In the case of high temperatures (we introduce the notation $\tau \equiv 1/\epsilon$), we obtain

$$\sigma(t) = U(t) = \frac{\hbar^2}{4}\{1 + 2[\lambda(\delta + \frac{1}{\delta})\tau + \mu(\delta - \frac{1}{\delta})\tau - 2\lambda]t\} \tag{33}$$

and the time when thermal fluctuations overtake quantum fluctuations is

$$t_d = \frac{\hbar\omega}{4kT[\lambda(\delta + \frac{1}{\delta})\tau + \mu(\delta - \frac{1}{\delta})\tau]}. \tag{34}$$

For $\delta = 1$ we obtain from Eq. (33)

$$\sigma(t) = U(t) = \frac{\hbar^2}{4}\{1 + 4\lambda(\tau - 1)t\}, \tag{35}$$

independent of μ and

$$t_d = \frac{\hbar\omega}{8kT\lambda}. \tag{36}$$

d) $r \neq 0$: At short times ($\lambda t \ll 1, \Omega t \ll 1$), we obtain from Eq. (23):

$$\sigma(t) = \frac{\hbar^2}{4}\{1 + 2[\lambda(\delta + \frac{1}{\delta(1-r^2)})\coth\epsilon + \mu(\delta - \frac{1}{\delta(1-r^2)})\coth\epsilon - 2\lambda]t\}. \tag{37}$$

The time when thermal fluctuations become comparable with quantum fluctuations is in this case

$$t_d = \frac{1}{2[\lambda(\delta + \frac{1}{\delta(1-r^2)})\coth\epsilon + \mu(\delta - \frac{1}{\delta(1-r^2)})\coth\epsilon - 2\lambda]}. \tag{38}$$

i) At zero temperature $T = 0$, the uncertainty becomes ($\mu = 0$):

$$\sigma_0(t) = \frac{\hbar^2}{4}\{1 + 2\lambda(\delta + \frac{1}{\delta(1-r^2)} - 2)t\} \tag{39}$$

and

$$t_d = \frac{1}{2\lambda(\delta + \frac{1}{\delta(1-r^2)} - 2)}. \tag{40}$$

ii) At high temperature

$$\sigma(t) = \frac{\hbar^2}{4}\{1 + 2[\lambda(\delta + \frac{1}{\delta(1-r^2)})\tau + \mu(\delta - \frac{1}{\delta(1-r^2)})\tau - 2\lambda]t\} \tag{41}$$

and the time when thermal fluctuations overtake quantum fluctuations is

$$t_d = \frac{\hbar\omega}{4kT[\lambda(\delta + \frac{1}{\delta(1-r^2)})\tau + \mu(\delta - \frac{1}{\delta(1-r^2)})\tau]}. \tag{42}$$

The expresions (29) of the uncertainty $\sigma = U$ (for $r = 0$) and (37) of the uncertainty σ (for $r \neq 0$) for short initial times evidently fulfil (by virtue of the condition (18)) the uncertainty principle. From the expressions (30) and (38) we observe that the time when the thermal fluctuations become comparable with the quantum fluctuations is decreasing with the increasing of both temperature T and dissipation λ.

5 Discussion of results

a) One often regards the regime where thermal fluctuations begin to surpass quantum fluctuations as the transition point from quantum to classical statistical mechanics and identifies the high temperature regime of a system as the

classical regime. On the other hand, it is known that a necessary condition for a system to behave classically is that the interference terms in its wave function have to diminish below a certain level, so that probability can be assigned to classical events [3, 4]. This is the decoherence process. The decoherence via interaction with an environment views the disappearance of the off-diagonal components of a reduced density matrix in some special basis as signaling a transition from quantum to classical physics. In Refs. [1, 2, 3, 4] it was shown that these two criteria of classicality are equivalent: the time when the quantum system decoheres is also the time when thermal fluctuations overtake quantum fluctuations. However the regime after thermal fluctuations dominate should not be called classical. After the decoherence time, although the system is describable in terms of probabilities, it cannot yet be regarded as classical because of the spin-statistics effects and has to be described by non-equilibrium quantum statistical mechanics. Only after the relaxation time the system can be correctly described by the equilibrium quantum statistical mechanics. The classical regime starts at a much later time. Only at a sufficiently high temperature when the spin (Fermi-Dirac or Bose-Einstein) statistics can be represented by the Maxwell-Boltzmann distribution function, can the system be considered in a classical regime (see Eqs. (26) – (28)).
b) The case of zero coupling, $\lambda = 0$ and $\mu = 0$, corresponds to an isolated harmonic oscillator taken as a closed quantum system. We find the Heisenberg quantum uncertainty function for an initial squeezed state to be (see Eq. (20))

$$U(t) = \frac{\hbar^2}{4}\{1 + \frac{1}{4}(\delta - \frac{1}{\delta})^2 \sin^2(2\omega t)\} \geq \frac{\hbar^2}{4}. \tag{43}$$

This is the quantum uncertainty relation for squeezed states. As the coupling to the environment goes to 0, the thermal fluctuations go also to 0 and the time-dependent term is the result of quantum fluctuations only. For the unsqueezed coherent state, $\delta = 1$, we recover the Heisenberg uncertainty relation $U(t) = \hbar^2/4$. For the same case of zero coupling, the Schrödinger uncertainty function (23) becomes $\sigma(t) = \hbar^2/4$ for any correlated coherent initial state.
c) In all expressions for the uncertainty functions obtained in the preceding Sec. 4, the terms depending on t are functions of the initial spread and correlation coefficient and represent the initial growth of thermal fluctuations, starting from the pure quantum fluctuations at $t = 0$. Using condition (18), we notice that for short initial times the uncertainty increases with dissipation λ and temperature T. This is in contrast with other models studied in literature [1, 2, 3, 4], where the uncertainty principle is violated on a short time scale as a consequence of the well-known violation of the positivity of the density operator [2, 15]. In fact, the uncertainty principle in our model is fulfilled not only for short times, but for any time and temperature and for the full range of the squeezing and correlation parameters. Indeed, cf. Eq. (9), by virtue of the complete positivity property of the Lindblad model, the generalized uncertainty function σ always fulfills the uncertainty principle and for $U(\geq \sigma)$ this is also true.

The time dependence of the uncertainties $U(t)$ and $\sigma(t)$ given by the dissipative terms reflects the fact that the Lindblad evolution of the system is non-unitary and is an expression of the effect of the environment. This is in contrast with the usual Liouville-von Neumann unitary evolution, when the uncertainty is independent of time, being invariant under unitary transformations.

d) In the high temperature limit, the time t_d when thermal fluctuations overtake quantum fluctuations obtained in our model is of the same scale as the decoherence time. The value of this time was determined in a series of papers in quantum Brownian motion models for initial coherent states [16, 17].
e) The second time scale of importance is the relaxation time scale, $t_{rel} = \lambda^{-1} \gg t_d$, when the particle reaches equilibrium with the environment. After this time, the uncertainty function takes on the Bose-Einstein form (26). At high temperatures the system reaches the Maxwell-Boltzmann limit and the uncertainty function takes on the classical form (28).
f) In the case of zero temperature, there are no longer thermal fluctuations and the environmentally induced fluctuations are of quantum nature only, given by terms describing both decay and oscillatory behaviour in the case of the uncertainty function U (22) and only decay behaviour in the case of the uncertainty function σ (25).

6 Summary and concluding remarks

In the present paper we have studied the evolution of the one-dimensional harmonic oscillator with dissipation within the framework of the Lindblad theory for open quantum systems. We have considered the case of an environment consisting of a thermal bath at an arbitrary temperature. The Gaussian correlated coherent, squeezed and Glauber coherent states were taken as initial states. We have derived closed analytical expressions of the Heisenberg and Schrödinger uncertainty functions for the evolution of the damped harmonic oscillator for different regimes of time and temperature, in particular in the limiting cases of both zero temperature and high temperature of the environment, in the limit of short times and long times and in the limit of zero coupling between the system and environment. Besides the dissipation constant these expressions give the explicit dependence on the squeezing parameter and the correlation coefficient. The obtained uncertainty functions show explicitly the contributions of quantum and thermal fluctuations of the system and environment. There are three contributions to the uncertainty: i) uncertainty which is intrinsic to quantum mechanics, expressed through the Heisenberg uncertainty principle (27), which is not dependent on the dynamics; ii) uncertainty that arises due to the spreading or reassembly (the reverse of spreading) of the wave packet, which depends on the dynamics and may increase or decrease the uncertainty; iii) uncertainty due to the coupling to a thermal environment, which has two components: dissipation and diffusion (this latter is responsible for the process of decoherence); this generally tends to increase the uncertainty as time evolves. In the Lindblad model the uncertainty relations are fulfilled, while in some other models considered in literature, the uncertainty relations are violated at some initial moments of time.

We have described the evolution of the system from a quantum pure state to a non-equilibrium quantum statistical state and to an equilibrium quantum statistical state and we have analyzed the relaxation process. We also found the regimes in which each type of fluctuations is important. The three stages are marked by the decoherence time and the relaxation time, respectively. The regime in which thermal fluctuations become comparable with the quantum fluctuations coincides with the regime in which the decoherence effects come into play. In other words, the system evolves from a quantum-dominated state to a thermal-dominated state in a time which is comparable with the decoherence time calculated in the context of quantum to classical transitions [1, 2, 3, 4].

With this study one can understand the relation between quantum, thermal and classical fluctuations. With the two characteristic times, namely the relaxation time and the decoherence time to be determined in further studies on environment-induced decoherence in the Lindblad model, one can give further contributions in describing the role of quantum and thermal fluctuations and, using the uncertainty relations, the transition from quantum to classical physics. In this context we have shown recently [12, 18] that in the Lindblad model the Schrödinger generalized uncertainty relation is minimized for all times for Gaussian pure initial states of the form of correlated coherent states for a special choice of the diffusion and dissipation coefficients. Such states are therefore the ones that suffer the least amount of noise and they are connected with the decoherence phenomenon [17, 18], being the most predictable and stable under the evolution in the presence of the environment.

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