Analytical solutions of a Dirac bound state equation and their relativistic interpretation

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Abstract

We solve the single particle Dirac equation with a particular confining potential and comment its relativistic significance. We show that the solutions describe a complex physical system made of independent constituents: a free particle and an effective field representing the confining potential.

1 The Dirac equation

In the description of low energy hadronic processes the so called "QCD inspired" models based on phenomenological notions of constituent quarks, confining potentials and hyperfine interaction, have been extensively used in connection with the problem of a light quark bound to a heavy antiquark [1]. In order to understand the connection of this kind of models with the real QCD a first requirement is to express in field language the information aquired with their aid.

In a recent paper [2] we have presented a way to do this. We have shown that in the momentum space representation a bound system of particles can be treated like a gas of free particles and a collective excitation of a background field which must be seen as the stationary time averaged result of a continuous series of quantum fluctuations.

The main features of the method are now discussed on an example which can be solved exactly. The solutions are particularly simple and have some nice features which deserve a detailed examination.

We consider the case of a single particle with spin 1/2 confined to a certain region of space by an external field represented by a particular scalar-vector combination of linear rising and Coulomb-like potentials frequently used in quark

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models [3]. The bound state function, ψ, is an eigenfunction of the single particle Dirac Hamiltonian and satisfies the equation

$$(\vec{\alpha} \cdot \vec{p} + \beta m + \mathcal{V}(\vec{r}))\, \psi(\vec{r}) = \varepsilon\ \psi(\vec{r}) \tag{1}$$

where

$$\mathcal{V}(\vec{r}) = \mathcal{V}_{\pm}(\vec{r}) = \frac{1}{2}\beta\, (\mathcal{V}_1(\vec{r}) + \mathcal{V}_2(\vec{r})) \pm \frac{1}{2}\, (\mathcal{V}_1(\vec{r}) - \mathcal{V}_2(\vec{r})) \tag{2}$$

and

$$\mathcal{V}_1(\vec{r}) = \zeta\ |\vec{r}| \quad \text{and} \quad \mathcal{V}_2(\vec{r}) = \frac{\xi}{|\vec{r}|} - 2m_i + 2\sqrt{\frac{\zeta}{\xi}}\ \left(\vec{\sigma} \cdot \vec{L} + \frac{1}{2}\right). \tag{3}$$

It is worthwhile mentioning that exact solutions of the single particle problem have been found also for scalar-vector oscillator potential [4] and for other combinations of scalar-vector Coulomb-like and linear rising potentials [5].

Turning now to eq.(refh1) and following the usual treatment of the Dirac equation with a central potential [6] we write ψ as:

$$\psi(\vec{r}) = \frac{1}{r} \left(\begin{array}{c} F(r)\ \mathcal{Y}^M_{lJ} \\ iG(r)\ \mathcal{Y}^M_{l'J} \end{array} \right) \tag{4}$$

where $l,\ l'$ are either $l = J - \frac{1}{2},\ l' = J + \frac{1}{2}$, or $l = J + \frac{1}{2},\ l' = J - \frac{1}{2}$ and $\mathcal{Y}^M_{lJ} = \sum_{m,s}\ C^{l\ \frac{1}{2}\ J}_{m\ s\ M}\ Y^m_l(\frac{\vec{r}}{r})\ \chi^s$.

Taking $\mathcal{V}^{(i)} = \mathcal{V}_+$ in (1) and recalling the relations

$$\vec{\alpha} \cdot \vec{p} = \alpha_r\ \left[p_r + \frac{i}{r}\ (\vec{J}^2 - \vec{L}^2 + \frac{1}{4})\right] \tag{5}$$

and

$$\frac{\vec{\sigma} \cdot \vec{r}}{r} \mathcal{Y}^M_{(J\pm\frac{1}{2})\ J} = -\mathcal{Y}^M_{(J\mp\frac{1}{2})\ J} \tag{6}$$

where $\alpha_r = \frac{\vec{r}}{r}\vec{\alpha}$, $p_r = -i\frac{1}{r}\frac{\partial}{\partial r}r$, $\vec{J}$ is the total angular momentum, $\vec{L}$ is the orbital momentum we replace ec.(1) by two coupled differential equations

$$-G'(r) + \frac{J(J+1) - l'(l'+1) + \frac{1}{4}}{r}\ G(r)\ =\ (\varepsilon - m - \zeta r)\ F(r) \tag{7}$$

$$\begin{aligned} & F'(r) - \frac{J(J+1) - l(l+1) + \frac{1}{4}}{r}\ F(r) = \\ & = \left[\varepsilon - m + \frac{\xi}{r} + 2\sqrt{\frac{\zeta}{\xi}}\left(J(J+1) - l'(l'+1) - \frac{1}{4}\right)\right]\ G(r). \end{aligned} \tag{8}$$

Now we consider the case $l = J - \frac{1}{2},\ l' = J + \frac{1}{2}$ and observe that a first condition for ψ to be finite at the origin is $F(r) \sim r^{1+k}, k \geq 0$. From (7) it follows $G(r) \sim r^{2+k}$ and from (8) one gets $k = l$. We also observe from (8) that $G'(r)$ behaves like $rF(r)$ at infinity, so we take

$$F(r) = r^{l+1} \mathrm{e}^{-\alpha r} \sum_{i=0} a_i\ r^i \tag{9}$$

$$G(r) = r^{l+2} \mathrm{e}^{-\alpha r} \sum_{i=0} b_i\ r^i . \tag{10}$$

Introducing the expressions (9) and (10) in the equations (7) and (8) we get the following relations connecting the coefficients $a_i,\ b_i$:

$$-(2l+n+3)b_n + \alpha b_{n-1} = (\varepsilon - m)a_n - \zeta a_{n-1} \tag{11}$$

$$na_n - \alpha a_{n-1} = \left[\varepsilon - m - 2\sqrt{\frac{\zeta}{\xi}} \left(l + \frac{3}{2} \right) \right] b_{n-2} + \xi b_{n-1} . \tag{12}$$

In order to cut the series in (9) at $n = N$ we require $a_{N+k} = 0$ for any $k \geq 1$ obtain from (12) the following restrictions for b_{N+k}:

$$0 = \left[\varepsilon - m - 2\sqrt{\frac{\zeta}{\xi}} \left(l + \frac{3}{2} \right) \right] b_{N+k-1} + \xi b_{N+k} . \tag{13}$$

Equation (11) gives for any $k \geq 1$:

$$-(2l+N+k+1)b_{N+k} + \alpha b_{N+k-1} = -\delta_{k1}\ \zeta a_{N+k-1} . \tag{14}$$

We observe that if the series in $G(r)$ is cut at $i = M$ where $M \geq N+1$, it follows from (14) that $b_i = 0$ for any $N \ < i \leq M$. On the other side, the condition $b_N \neq 0$ implies

$$\varepsilon - m - 2\sqrt{\frac{\zeta}{\xi}} \left(l + \frac{3}{2} \right) = 0 \tag{15}$$

which is the quantification condition for the energy levels. Introducing the condition (15) in the equations (11) and (12) we write them as follows

$$-(2l+n+3)b_n + \alpha b_{n-1} = 2\sqrt{\frac{\zeta}{\xi}} \left(l + \frac{3}{2} \right) a_n - \zeta a_{n-1} \tag{16}$$

$$(n+1)a_{n+1} - \alpha a_n = \xi b_n . \tag{17}$$

For $n = 0, 1, 2, ..., N$ eqs.(16) and (17) give

$$\begin{aligned} &-(2l+3)b_0 = 2\sqrt{\frac{\zeta}{\xi}} \left(l + \frac{3}{2} \right) a_0 \\ &a_1 - \alpha a_0 = \xi b_0 \end{aligned}$$

$$
\begin{aligned}
&-2(l+2)b_1 + \alpha b_0 = 2\sqrt{\frac{\zeta}{\xi}}\left(l+\frac{3}{2}\right)a_1 - \zeta a_0 \\
&2a_2 - \alpha a_1 = \xi b_1 \\
&-\alpha a_N = \xi b_N \\
&\alpha b_N = -\zeta a_N.
\end{aligned}
\tag{18}
$$

From the last two equalities it follows that

$$
\alpha = \sqrt{\zeta\xi}; \qquad \zeta\xi > 0. \tag{19}
$$

Introducing these relations into the first equalities of the set (18) we have successively: $a_1 = 0,\ b_1 = 0,\ a_2 = 0, ...$ which means that the solution of the eigenvalue equation (1) reads

$$
\psi_J^M(\vec{r}) = \begin{pmatrix} r^{J-\frac{1}{2}} \mathrm{e}^{-\sqrt{\zeta\xi}r}\ \mathcal{Y}^M_{(J-\frac{1}{2})J} \\ -i\sqrt{\frac{\zeta}{\xi}}\ r^{J+\frac{1}{2}} \mathrm{e}^{-\sqrt{\zeta\xi}r}\ \mathcal{Y}^M_{(J+\frac{1}{2})J} \end{pmatrix} \tag{20}
$$

and $\varepsilon_J = m + 2\sqrt{\frac{\zeta}{\xi}}(J+1)$.

Proceeding in a similar manner it can be shown that there is no solution in the case $l = J + \frac{1}{2},\ l' = J - \frac{1}{2}$, so (20) is the single one and ε_J is $(2J+1)$ fold degenerate.

We also define $\bar{\psi}_J^M = (\psi_J^M)^+\gamma^0$ which satisfies the same equation as ψ_J^M and the charge conjugate solution ψ_J^{cM}

$$
\psi_J^{cM}(\vec{r}) = i\gamma^2\gamma^0(\bar{\psi}_J^M)^T(\vec{r}) = \begin{pmatrix} i\sqrt{\frac{\zeta}{\xi}}\ r^{J+\frac{1}{2}} \mathrm{e}^{-\sqrt{\zeta\xi}r}\ \mathcal{Y}^M_{(J+\frac{1}{2})J} \\ r^{J-\frac{1}{2}} \mathrm{e}^{-\sqrt{\zeta\xi}r}\ \mathcal{Y}^M_{(J-\frac{1}{2})J} \end{pmatrix} \tag{21}
$$

which satisfies the equation (1) with the potential $\mathcal{V}_-$ and corresponds to the eigenvalue $\varepsilon_J^c = -\varepsilon_J = -m - 2\sqrt{\frac{\zeta}{\xi}}(J+1)$.

2 The relativistic interpretation of the bound state function

Our purpose now is to find the relativistic meaning of the bound state functions ψ and ψ^c in the hope to create a link with the quantum field theory.

To this end we first expand ψ and ψ^c in terms of the free Dirac solutions, the only ones having a real relativistic character and then look for a Lorentz covariant interpretation of the contribution of the confining potential to the bound state energy [2].

For reasons of simplicity we work with the lowest J functions

$$
\psi^\rho_{\frac{1}{2}}(\vec{r}) = \begin{pmatrix} \mathrm{e}^{-\sqrt{\zeta\xi}r}\chi^\rho \\ -i\sqrt{\frac{\zeta}{\xi}}\ r\mathrm{e}^{-\sqrt{\zeta\xi}r}\mathcal{Y}^\rho_{1\frac{1}{2}} \end{pmatrix} \tag{22}
$$

and

$$\psi^{c\rho}_{\frac{1}{2}}(\vec{r}) = \begin{pmatrix} i\sqrt{\frac{\zeta}{\xi}}\ \mathrm{r}\mathrm{e}^{-\sqrt{\zeta\xi}r}\ \mathcal{Y}^{\rho}_{1\frac{1}{2}} \\ \mathrm{e}^{-\sqrt{\zeta\xi}r}\ \varphi^{\rho} \end{pmatrix} \tag{23}$$

where χ^ρ and φ^ρ are two component spinors and project them on the free Dirac spinors.

In the first case the projection $\phi^{(+)}_{\rho s}(\vec{k})$ on a positive energy free state has the following expression

$$\begin{aligned} \phi^{(+)}_{\rho s}(\vec{k}) &= \int d^3r\mathrm{e}^{-i\vec{k}\vec{r}}\bar{u}^s(\vec{k})\psi^{\rho}_{\frac{1}{2}}(\vec{r}) = \\ &= \frac{4\pi\sqrt{\zeta\xi}}{(\zeta\xi+\vec{k}^2)^2}\frac{\sqrt{e+m}}{\sqrt{2m^3}}\left(1-4\sqrt{\frac{\zeta}{\xi}}\frac{e-m}{\zeta\xi+\vec{k}^2}\right)\delta_{\rho s} \end{aligned} \tag{24}$$

where $e=\sqrt{\vec{k}^2+m^2}$ and, according to the general principles of the quantum mechanics, it represents the probability amplitude to find a free particle with momentum $\vec{k}$ and spin s in the bound state $\psi^{\rho}_{\frac{1}{2}}$.

Projecting ψ on a negative energy free Dirac state one obtains

$$\begin{aligned} \phi^{(-)}_{\rho s}(\vec{k}) &= \int d^3r\mathrm{e}^{i\vec{k}\vec{r}}\bar{v}^s(\vec{k})\ \psi^{\rho}_{\frac{1}{2}}(\vec{r}) = \\ &= \frac{4\pi\sqrt{\zeta\xi}}{(\zeta\xi+\vec{k}^2)^2}\frac{1}{\sqrt{2m(e+m)}}\left(1+4\sqrt{\frac{\zeta}{\xi}}\frac{e+m}{\zeta\xi+\vec{k}^2}\right)\bar{v}^s(\vec{k})\gamma^0u^\rho(\vec{k}). \end{aligned} \tag{25}$$

We notice that the probability amplitude of a free particle with negative energy is proportional with $\bar{v}^s(\vec{k})\gamma^0u^\rho(\vec{k})$ which can be seen as the amplitude of dissolution into the vaccuum of a particle-antiparticle pair with equal momenta. We recall that the projections of negative energy free states on the free states with positive energy are allways zero and hence the existence of the nonvanishing result (25) has to be considered an effect of a classical potential. Our remark shows that ψ is not a single particle state in the sense of the classical quantum mechanics and that its physical content is more complex than that. We conclude that a complete relativistic description of a bound state is impossible without introducing the contribution of the confining potential in a way compatible with Lorentz covariance.

To this purpose we project eq.(1) on a free Dirac state with positive (negative) energy and momentum $\vec{k}$ and observe that the quantity Q^0 defined as

$$Q^0 = \varepsilon \mp \sqrt{\vec{k}_1^2+m_1^2} \tag{26}$$

is proportional with the matrix element of the confining potential between the free state and the bound state function ψ and hence we consider it is the energy associated with the center of forces.

Furthermore, taking into account that $\vec{r}=\vec{R}-\vec{R}_0$, where $\vec{R},\vec{R}_0$ are the position vectors of the particle and of the center of forces with respect to the origin of the coordinate frame, the bound state function ψ appears to describe a system made of two independent constituents: a free particle with momentum $\vec{k}$ and an effective component representing the center of forces. The last one, denoted Φ in the following, carries the momentum $\vec{Q}$ which is a recoil momentum due to the motion of the free particle and is defined as:

$$\vec{Q} = -\vec{k}. \tag{27}$$

It is worthwhile mentioning that by introducing Φ as an independent component of the bound state and by defining its 4-momentum as a linear combination of free 4-momenta (see (26) and (27)) the contribution of the binding potential to the bound state energy aquires a well defined, Lorentz covariant significance. The major advantage of this fact is that one can write immediately a Lorentz covariant expression of the bound state function without worrying about the transformation properties of the binding potential at boosts. This can be easily achieved with the aid of the vector π^μ which is directed along the time axis in the reference frame where the confining potential is defined. Its expression in a reference frame moving with the velocity $\vec{\omega}$ with respect to the first one is $\pi^\mu = (\frac{1}{\sqrt{1-\omega^2}}, \frac{1}{\sqrt{1-\omega^2}}\vec{\omega})$ and the single change to be made in the expressions of $\phi^{(\pm)}$ is to replace $\vec{p}$ by $p_T^\mu = p^\mu - (\pi \cdot p)\pi^\mu$.

Then we have:

$$\phi^{(+)}_{\rho s}(k) = \frac{4\pi\sqrt{\zeta\xi}}{(\zeta\xi + (\pi\cdot k)^2 - m^2)^2} \frac{\sqrt{(\pi\cdot k) - m}}{m\sqrt{2m}} \left(1 - 4\sqrt{\frac{\zeta}{\xi}} \frac{(\pi\cdot k) - m}{\zeta\xi + (\pi\cdot k)^2 - m^2}\right) \delta_{\rho s},$$

$$\phi^{(-)}_{\rho s}(k) = \frac{4\pi\sqrt{\zeta\xi}}{(\zeta\xi + (\pi\cdot k)^2 - m^2)^2} \left(\frac{1}{\sqrt{2m((\pi\cdot k) + m)}} \left(1 + 4\sqrt{\frac{\zeta}{\xi}} \frac{(\pi\cdot k) - m}{\zeta\xi + (\pi\cdot k)^2 - m^2}\right)\right.$$
$$\times \bar{v}^s(k)(\pi\cdot\gamma)u^\rho(k). \tag{28}$$

Introducing the quark creation and annihilation operators, the stationary Lorentz covariant expression of the bound state function ψ is:

$$\psi^\rho(\vec{R}, \vec{R}_0, t) =$$
$$\sum_\rho \int d^3k \frac{m}{e} d^4Q \, \left(\delta^{(4)}(Q + k - P)\; \phi^{(+)}_{\rho s}(k) a_s(k)\Phi(Q)\; u^s(k) \mathrm{e}^{-i(e+Q^0)t + i\vec{k}\vec{R} + i\vec{Q}\vec{R}_0}\right.$$
$$\left. -\delta^{(4)}(Q - k - P)\; \phi^{(-)}_{\rho s}(k) b^\dagger_s(k)\Phi(Q)\; v^s(k)\mathrm{e}^{i(e-Q^0)t - i\vec{k}\vec{R} + i\vec{Q}\vec{R}_0}\right) \tag{29}$$

where the annihilation operator of a negative energy free state a_- has been replaced by the creation operator of an antiparticle, $b^\dagger$. The four components of the momentum P^μ are $E = \frac{\varepsilon}{\sqrt{1-\omega^2}}$ and $\vec{P} = -\vec{\omega}E$. Φ designates the additional component representing the confining potential and, as it can be seen from (29), it must be seen as a reservoir of particles and energy in the sense the vaccuum is for the free Dirac equation.

Similar results are obtained in the case of the charge conjugate function where one writes

$$\psi^{c\rho}(\vec{R}, \vec{R}_0, t) =$$

$$\sum_{\rho} \int d^3k \frac{m}{e} d^4Q \ \Big(\delta^{(4)}(k+Q-P) \phi^{c(-)}_{\rho s}(k) a_s(k) \Phi(Q) \ u^s(k) \mathrm{e}^{-i(e+Q^0)t + i\vec{k}\vec{R} + i\vec{Q}\vec{R}_0}$$
$$-\delta^{(4)}(Q-P-k) \phi^{c(+)}_{\rho s}(k) b^{\dagger}_s(k) \Phi(Q) \ v^s(k) \mathrm{e}^{i(e-Q^0)t - i\vec{k}\vec{R} + i\vec{Q}\vec{R}_0} \Big) \tag{30}$$

where $\phi^{c(+)}$ and $\phi^{c(-)}$ can be obtained from $\phi^{(+)}$ and $\phi^{(-)}$ respectively by performing the replacement $u^r \leftrightarrow -v^r$.

Concluding this paper we notice that the relativistic representation of a particle in a bound state is that of a system made of two independent components: a free particle and an effective excitation of some background field, Φ, which has a double face: in momentum space it is a reservoir of particles and energy while in coordinate representation it represents a kind of a box where the particle is confined.

This image is similar to that of bag models [7]. It is a time averaged image, not an instantaneous one and it is expected to hold whenever the observation time is longer or at least equal to the time giving a stable average.

We also notice that, as suggested by the relativistic interpretation of the bound state functions (29) and (30), the Dirac equation with a confining potential is a field equation and its solutions represent a complex physical structure where the free positive and negative energy states are mixed.

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References

[1] M. G. Olsson, S. Veseli and K. Williams, Phys. Rev **D** 51 (1995) 5079; J. Sucher, Phys. Rev. **D** 51 (1995) 5965.

[2] L. Micu, **hep-ph/0202244**.

[3] W. Lucha, F. Schöberl and D. Gromes, Phys. Rep. **200** (1991) 127.

[4] R. Tegen, R. Brockmann and W. Weise, Z. Phys. A - Atoms and Nuclei **307** (1982) 339.

[5] J. Franklin, Mod. Phys. Lett. **A14** (1999) 2409; A. S. De Castro and J. Franklin, **hep-ph/0011137**; A. S. De Castro and J. Franklin, J. Mod. Phys. **A15:** (2000) 4355.

[6] M. E. Rose, Relativistic electron theory, John Wiley & Sons Inc. New York-London, 1960.

[7] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn and V. F. Weisskopf, Phys. Rev. D **9**, 3471 (1974); A. Chodos, R. L. Jaffe, K. Johnson and C. B. Thorn, Phys. Rev. D **10**, 2599 (1974); T. DeGrand, R. L. Jaffe, K. Johnson and J. Kiskis, Phys. Rev. D **12**, 2060 (1975).