An integral equation method for the inverse conductivity problem.

S. Ciulli[†], *M.K. Pidcock*[‡] *and C. Sebu*[†]

[†]*Laboratoire de Physique–Mathématique et Théorique, Université de Montpellier II, 34095 Montpellier, France;*

[‡]*Department of Mathematical Sciences, Oxford Brookes University, Oxford OX33 1HX, United Kingdom;*

Abstract: *We present an image reconstruction algorithm for the Inverse conductivity Problem based on reformulating the problem in terms of integral equations. By using a priori information we are able to find a regularized conductivity distribution by first solving a Fredholm integral equation of the second kind and then by solving a first order partial differential equation for the conductivity $\sigma(\mathbf{x})$. Many of the calculations involved in the method can be achieved analytically using the eigenfunctions of an integral operator defined in the paper.*

1. Introduction

It has long been recognized that the inverse conductivity problem, namely that of deducing the exact form of the electrical conductivity $\sigma(\mathbf{x})$ of an object Ω using electrical measurements performed only on its boundary $\partial\Omega$, is an extremely ill-posed, non-linear inverse problem. However owing to its practical importance in geophysics, medicine or industry in general, a very large scientific literature has been devoted to this subject. Indeed, if the various type of matter within the object have widely differing electrical properties, the solution of this inverse problem should lead to images of the object's internal physical structure, without using any invasive or destructive procedures.

A pioneer work in this field has been that of Sabba Stefanescu, C. Schlumberger and M. Schlumberger in the early thirties [1], when these authors were preparing their PhD Thesis at Ecole Normale Supérieure in Paris. Although at that time there was little information about the dangers related to the Ill Posed Problems, their method was good enough to find most of the Texas oil. However years later Sabba Stefanescu commented [2] that while this method worked quite well when one had to find the depth at which lies the surface of some liquid, oil or water, often gave aberrant results when used to determine, for instance, the position or the shape of a copper vein. The present–day interpretation of these facts would be that while in the one parameter case the whole Banach space reduces to the one dimensional $\mathbb{R}^1$ space, the functions describing an ore deposit might be concealed in the very far dimensions of the infinite dimensional Banach space under consideration.

In the last twenty years or so there has been a renewed strong interest in determining the class of conductivity distributions that can be recovered [3] — [8] and in the development of related reconstruction algorithms. This interest has been generated by both interesting theoretical challenges and by the many new practical applications of this problem. The algorithms that have been proposed include the well-known back-projection methods originally developed by Barber and Brown [9] and a wide range of iterative algorithms based on formulating the inverse problem as a non linear optimization problem. These techniques are quite demanding computationally particularly when we address the full three dimensional problem where the number of discretisation points used in the numerical implementation of the algorithms becomes very large. This concern has encouraged the search for reconstruction algorithms which reduce the computational demands either by using some *a priori* information [10]— [13] or by developing analytic procedures.

In a recent paper [14] we presented a contribution to this effort based on a reformulation of the inverse problem in terms of integral equations. One feature of this work is that many of the calculations involve working with analytical expressions written in terms of the known eigenfunctions of the kernel of these equations, the computational part being restricted of the introduction of the data, the numerical evaluation of some of the analytic formulæ and the solution of a final integral equation. However, another feature of this first version of our method is that it requires the knowledge on the boundary of the region Ω not only of the electrical potential Φ and its normal derivative $\partial\Phi/\partial n$, but also of the electrical conductivity σ and its normal derivative $\partial\sigma/\partial n$. Although it is sometimes possible to measure the derivative of the conductivity, for example in geophysics, in other cases it is more difficult, especially in medical applications. In addition it represents a demand which is not present in the usual formulation of this inverse problem and so one could ask whether the problem was overdetermined. In fact it is not, since only specific combinations of these data are used, but this suggests that there might be formulations of our method in which the boundary values of the derivative of the conductivity are not required.

In this paper we describe a new version of our approach which does not require this additional information but uses only the data which is usually assumed to be available in this problem. A significant consequence of this new approach is that the final step in the reconstruction algorithm involves the solution of a linear first order partial differential equation rather than the second order one obtained in our previous work.

Throughout this paper we shall assume that the conductivity distribution σ is sufficiently smooth, with first order derivatives which are continuous in $\overline{\Omega}$. Such functions are dense in the larger class of the functions which have these properties only piecewise and which contains many of the physically interesting situations. In addition we shall assume that the domain Ω is a bounded open set whose boundary $\partial\Omega$ is sufficiently smooth, namely of class C^2.

2. The formulation of the problem.

For a general isotropic conductivity distribution $\sigma(\mathbf{x})$ the equation for the potential $\Phi(\mathbf{x})$ reads :

$$\nabla \cdot [\,\sigma(\mathbf{x}) \nabla \Phi(\mathbf{x})\,] = 0\,, \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^n\,, \quad n = 2 \text{ } or \text{ } 3\,. \tag{1}$$

The inverse conductivity problem can be stated as follows: from the values of $\sigma(\mathbf{x})$, of $\Phi(\mathbf{x})$ and of its normal derivative $\partial\Phi/\partial n$ measured on the boundary $\partial\Omega$, find the conductivity $\sigma(\mathbf{x})$ everywhere in Ω.

If we define $\tilde{\sigma}(\mathbf{x}) \equiv ln(\sigma(\mathbf{x}))$, equation (1) can be rewritten as

$$\nabla^2 \Phi(\mathbf{x}) = -Y(\mathbf{x})\,, \quad \text{where } \mathrm{Y}(\mathbf{x}) \equiv \nabla\tilde{\sigma}(\mathbf{x}) \cdot \nabla\Phi(\mathbf{x})\,. \tag{2}$$

Using Green's formula, this differential equation can be transformed into an equivalent pair of integral equations

$$\Phi(\mathbf{x}) = \chi_D(\mathbf{x}) + \int_\Omega d^n\mathbf{y}\, \mathcal{G}_D(\mathbf{x},\mathbf{y})\, Y(\mathbf{y}) \tag{3}$$

$$\Phi(\mathbf{x}) = \chi_N(\mathbf{x}) + \int_\Omega d^n\mathbf{y}\, \mathcal{G}_N(\mathbf{x},\mathbf{y})\, Y(\mathbf{y})\,, \tag{4}$$

where $\mathcal{G}_D$ and $\mathcal{G}_N$ are the Dirichlet and Neumann Green's functions for Laplace's equation in Ω ($\mathcal{G}_D(\mathbf{x},\mathbf{y})$ is zero if $\mathbf{x} \in \partial\Omega$ while $\partial\mathcal{G}_N(\mathbf{x},\mathbf{y})/\partial n|_{\partial\Omega}$ is constant). Further χ_D and χ_N are the two harmonic functions constructed respectively from the Dirichlet $\Phi(\mathbf{z})|_{\partial\Omega}$ and the Neumann $\partial\Phi/\partial n|_{\partial\Omega}$ boundary values of the potential:

$$
\begin{aligned}
\chi_D(\mathbf{x}) &= -\int_{\partial\Omega} d^{n-1}\mathbf{z}\, \frac{\partial \mathcal{G}_D}{\partial n}(\mathbf{x},\mathbf{z})\;\; \Phi(\mathbf{z})|_{\partial\Omega} \\
\chi_N(\mathbf{x}) &= \int_{\partial\Omega} d^{n-1}\mathbf{z}\, \mathcal{G}_N(\mathbf{x},\mathbf{z})\;\; \frac{\partial\Phi}{\partial n}\Big|_{\partial\Omega} + \frac{C}{|\,\partial\Omega\,|}
\end{aligned}
$$

where C is a constant equal to the value of the integral of χ_N over the boundary $\partial\Omega$. These two harmonic functions are different, unless $\sigma(\mathbf{x})$ is a constant.

Conversely, from the eqs. (3) and (4) and the properties of the Green's functions one can deduce both the initial equation $\nabla^2\Phi(\mathbf{x}) = -Y(\mathbf{x})$ and the boundary values $\Phi(\mathbf{z})|_{\partial\Omega}$ and $\partial\Phi/\partial n|_{\partial\Omega}$ of the potential Φ.

3. The Direct Problem.

Before considering the Inverse Problem, let us see how one might compute the potential Φ if the conductivity σ is known (i.e. the 'direct problem'). One way is to use the Finite Element Method to solve equation (1), but there are a variety of other ways of achieving this goal. For instance, as shown in [14], one may use the well known change of variables $\tau = \sqrt{\sigma}$ to transform equation (1) into an integral equation for the function $\Psi \equiv \tau\Phi$. Another way which will be discussed below, consists of applying the operator $\nabla_{\mathbf{x}}\tilde{\sigma}(\mathbf{x}) \cdot \nabla_{\mathbf{x}}$ to equation (3) to give

$$
Y(\mathbf{x}) = \nabla_{\mathbf{x}}\tilde{\sigma}(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\chi_D(\mathbf{x}) + \int_{\Omega} d^n\mathbf{y}\, \nabla_{\mathbf{x}}\tilde{\sigma}(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\mathcal{G}_D(\mathbf{x},\mathbf{y})\, Y(\mathbf{y})\,. \tag{5}
$$

This is an integral equation for the Laplacian of Φ. Once $Y(\mathbf{x})$ is known, one can compute $\Phi(\mathbf{x})$ by means of a simple quadrature using formula (3).

Not all integral equations can be processed numerically without some preparation. It would have been nice for the practical solution of equation (5) if the integral operator defined by the kernel $K(\mathbf{x},\mathbf{y}) = \nabla_{\mathbf{x}}\tilde{\sigma}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\mathcal{G}_D(\mathbf{x},\mathbf{y})$ were compact, since compact operators imply the Fredholm Alternative [15]. The Hilbert–Schmidt condition $K(.,.) \in L^2(\Omega\times\Omega)$ is sufficient for the compactness of the integral operator. However $\nabla_{\mathbf{x}}\mathcal{G}_D(\mathbf{x},\mathbf{y})$ has a singularity of the type $1/r$ in the two dimensional case and of the type $1/r^2$ in three dimensions, and hence in general $\int_{\Omega} d^n\mathbf{x} \int_{\Omega} d^n\mathbf{y}\; |\nabla_{\mathbf{x}}\tilde{\sigma}(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\mathcal{G}_D(\mathbf{x},\mathbf{y})|^2$ is divergent. This means that we cannot deduce that the integral equation (5) is of Fredholm type and that it can be solved using standard procedures. However, we shall show below that the situation can be recovered by considering its first iteration.

Theorem

For conductivities which are C^1 on Ω, the kernel $K_2(\mathbf{x},\mathbf{z}) = \int_{\Omega} d^n\mathbf{y}\, K(\mathbf{x},\mathbf{y})K(\mathbf{y},\mathbf{z})$ of the first iterated equation (see below eq.(10)) of equation (5) is Hilbert-Schmidt.

Proof

We shall consider here the three dimensional case but the proof in two dimensions is similar. Namely we shall prove that for a given conductivity distribution σ in $\Omega \in \mathbb{R}^3$ the first iterated kernel

$$
K_2(\mathbf{x},\mathbf{z}) \equiv \int_{\Omega} d^3\mathbf{y}\, \nabla_{\mathbf{x}}\tilde{\sigma}(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\mathcal{G}_D(\mathbf{x},\mathbf{y}) \nabla_{\mathbf{y}}\tilde{\sigma}(\mathbf{y}) \cdot \nabla_{\mathbf{y}}\mathcal{G}_D(\mathbf{y},\mathbf{z}) \tag{6}
$$

is Hilbert-Schmidt. For smooth conductivities, any singular behaviour in the integrand of equation (6) comes from the gradient of the Dirichlet Green's function. If we consider just the leading singularity we need to retain only terms of the form

$$\nabla_{\mathbf{x}}\mathcal{G}_D(\mathbf{x},\mathbf{y}) \sim \frac{1}{4\pi}\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\,.$$

Since $\widetilde{\sigma} \in C^1$, for any $\epsilon > 0$ there exists a ball $\Omega_R \subset \Omega$ of radius R with centre $\mathbf{x}$ such that

$$|\nabla_{\mathbf{x}}\widetilde{\sigma}(\mathbf{x}) - \nabla_{\mathbf{y}}\widetilde{\sigma}(\mathbf{y})| < \epsilon$$

for all $\mathbf{y} \in \Omega_R$. Hence in this ball we can approximate $\nabla_{\mathbf{y}}\widetilde{\sigma}(\mathbf{y})$ by $\nabla_{\mathbf{x}}\widetilde{\sigma}(\mathbf{x})$.
With these considerations the most singular parts of $K_2(\mathbf{x},\mathbf{z})$ can be written as

$$K_2(\mathbf{x},\mathbf{z}) \sim K_2^1(\mathbf{x},\mathbf{z}) + K_2^2(\mathbf{x},\mathbf{z}) \tag{7}$$

where

$$K_2^1(\mathbf{x},\mathbf{z}) = \frac{1}{16\pi^2}\int\limits_{\Omega\backslash\Omega_R} d^3\mathbf{y}\, \nabla_{\mathbf{x}}\widetilde{\sigma}(\mathbf{x})\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\nabla_{\mathbf{y}}\widetilde{\sigma}(\mathbf{y})\cdot\frac{(\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|^3}$$

and

$$K_2^2(\mathbf{x},\mathbf{z}) = \frac{1}{16\pi^2}\int\limits_{\Omega_R} d^3\mathbf{y}\, \nabla_{\mathbf{x}}\widetilde{\sigma}(\mathbf{x})\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\nabla_{\mathbf{x}}\widetilde{\sigma}(\mathbf{x})\cdot\frac{(\mathbf{y}-\mathbf{z})}{|\mathbf{y}-\mathbf{z}|^3}$$

Since the modulus of the gradient of $\widetilde{\sigma}$ is bounded on Ω, we have

$$\left|K_2^1(\mathbf{x},\mathbf{z})\right| < \frac{Const}{16\pi^2}\int\limits_{\Omega\backslash\Omega_R} d^3y\,\frac{1}{|\mathbf{x}-\mathbf{y}|^2}\frac{1}{|\mathbf{y}-\mathbf{z}|^2}$$

When studying the limit $r = |\mathbf{x}-\mathbf{z}| \to 0$ we can take $r < R/N << R$. Then, for $\mathbf{y} \in \Omega\backslash\Omega_R$, it follows that $|\mathbf{y}-\mathbf{x}| > R$ and $|\mathbf{y}-\mathbf{z}| > R(1-1/N)$, and hence

$$\left|K_2^1(\mathbf{x},\mathbf{z})\right| < \frac{Const}{16\pi^2}\frac{|\Omega|}{(1-1/N)^2R^4}\,. \tag{8}$$

To evaluate $K_2^2(\mathbf{x},\mathbf{z})$ we define $\mathbf{x} = (0,0,0)$, $\mathbf{z} = (0,0,r)$ and $\mathbf{y} = (\rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta)$, where $(\rho,\theta,\phi) \in \mathbb{R}^3$ are local spherical coordinates in Ω_R. The constant gradient points in an arbitrary direction, so for illustration we shall consider $\nabla_{\mathbf{x}}\widetilde{\sigma}(\mathbf{x}) = (0,0,1)$. In this case

$$K_2^2(\mathbf{x},\mathbf{z}) \sim \frac{1}{8\pi}\int\limits_0^R d\rho\int\limits_0^\pi \sin\theta d\theta\frac{\cos\theta(\rho\cos\theta - r)}{(\rho^2 - 2\rho r\cos\theta + r^2)^{\frac{3}{2}}} = -\frac{1}{12\pi R}\,. \tag{9}$$

Combining equations (8) and (9) we see that $K_2(\mathbf{x},\mathbf{z})$ has no singularities when $|\mathbf{x}-\mathbf{z}| \equiv r \to 0$ — it is bounded — and hence Hilbert-Schmidt. ∎

This theorem implies that the first iteration of equation (5):

$$Y(\mathbf{x}) = \nabla_{\mathbf{x}}\widetilde{\sigma}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\chi_D(\mathbf{x}) + \int\limits_\Omega d^n\mathbf{y}\,K(\mathbf{x},\mathbf{y}))\nabla_{\mathbf{y}}\widetilde{\sigma}(\mathbf{y})\cdot\nabla_{\mathbf{y}}\chi_D(\mathbf{y}) + \int\limits_\Omega d^n\mathbf{y}\,K_2(\mathbf{x},\mathbf{y}))\,Y(\mathbf{y}) \tag{10}$$

can then be solved by means of the usual procedures for Fredholm equations. Once the function $Y(\mathbf{x})$ has been computed, the potential $\Phi(\mathbf{x})$ can be obtained by means of the formulæ (3) or (4). This procedure can be used to construct model functions against which we can study the performance of our method of solving the Inverse Problem.

4. The Inverse Problem

Our treatment of the inverse problem is initially similar to that described in [14]. Subtracting equation (3) from (4) we can find that

$$\chi(\mathbf{x}) - \int_{\Omega} d^n\mathbf{y}\, \mathcal{K}(\mathbf{x},\mathbf{y})\, Y(\mathbf{y}) = 0 \tag{11}$$

where

$$\begin{aligned}\mathcal{K}(\mathbf{x},\mathbf{y}) \equiv\ & \mathcal{G}_D(\mathbf{x},\mathbf{y}) - \mathcal{G}_N(\mathbf{x},\mathbf{y}) \quad \text{and} \\ & \chi(\mathbf{x}) = \chi_N(\mathbf{x}) - \chi_D(\mathbf{x})\,.\end{aligned}$$

Since $\mathcal{K}$ is the difference of two Green's functions it is harmonic and symmetric. Consequently it is Hilbert-Schmidt, which means that (11) is an ill-posed Fredholm equation of the first kind.

Since $\chi(\mathbf{x})$ can represent any $L^2(\partial\Omega)$–harmonic function, and since the null space of a self-adjoint Hilbert-Schmidt kernel is the orthogonal complement of its range, $Ker\mathcal{K}$ consists of all the functions which are orthogonal to the harmonic functions in Ω.

We can prove the following result concerning the potentials $\Phi_{\mathfrak{k}}$ related to this null space :

Theorem

If $\nabla^2\Phi_{\mathfrak{k}} = -Y_{\mathfrak{k}}$, where $Y_{\mathfrak{k}} \in Ker(\mathcal{K})$, then there exists a function $\Phi_{\mathfrak{k}}$ such that both its boundary values and that of its normal derivatives are identically zero

$$\Phi_{\mathfrak{k}}(\mathbf{x}) = 0,\ \ \frac{\partial\Phi_{\mathfrak{k}}}{\partial n}(\mathbf{x}) \ = 0\,,\ \ \mathbf{x} \in \partial\Omega\,. \tag{12}$$

In other words these functions are completely invisible to measurements of potentials and currents on the boundary.

Proof

Since the Laplacian of any harmonic function Φ_h is identically zero, it is clear that the $\Phi_{\mathfrak{k}}$ can be defined only up to an harmonic function Φ_h. Taking advantage of this fact, if the boundary values of some initial $\Phi'_{\mathfrak{k}}$ are not zero, we can always find another function $\Phi_{\mathfrak{k}} = \Phi'_{\mathfrak{k}} + \Phi_h$ so that

$$\Phi_{\mathfrak{k}}|_{\partial\Omega} = 0\,. \tag{13}$$

where

$$\Phi_h(\mathbf{x}) = +\int_{\partial\Omega} dy \frac{\partial\mathcal{G}_D(\mathbf{x},\mathbf{y})}{\partial n}\Phi'_{\mathfrak{k}}(\mathbf{y})\,. \tag{14}$$

has boundary values which are opposite to that of $\Phi'_{\mathfrak{k}}$. We apply now Green's formula

$$\int_{\Omega} d^2\mathbf{x}\, \Big(u(\mathbf{x})\,\nabla^2\Phi_{\mathfrak{k}}(\mathbf{x}) \,-\, \Phi_{\mathfrak{k}}(\mathbf{x})\nabla^2 u(\mathbf{x})\Big) \,=\, \int_{\partial\Omega} d\mathbf{x}\, \left(u(\mathbf{x})\,\frac{\partial\Phi_{\mathfrak{k}}(\mathbf{x})}{\partial n} \,-\, \Phi_{\mathfrak{k}}(\mathbf{x})\,\frac{\partial u(\mathbf{x})}{\partial n}\right)\,, \tag{15}$$

where $u(\mathbf{x})$ is an harmonic function. Since $\nabla^2 u = 0$ and $\nabla^2 \Phi_{\mathfrak{k}} = -Y_{\mathfrak{k}}$ is orthogonal to any harmonic function, the left hand side of the above equation is zero. But since (13) $\Phi_{\mathfrak{k}}|_{\partial\Omega} \equiv 0$, we see that

$$\int_{\partial\Omega} d\mathbf{x}\, u(\mathbf{x})\, \frac{\partial \Phi_{\mathfrak{k}}(\mathbf{x})}{\partial n} = 0\,. \tag{16}$$

To prove that the normal derivative of $\Phi_{\mathfrak{k}}$ is also identically zero on the boundary, we shall assume for the moment that for some given $x_0 \in \partial\Omega$,

$$\left.\frac{\partial \Phi_{\mathfrak{k}}(\mathbf{x})}{\partial n}\right|_{\mathbf{x}=\mathbf{x_0}} \neq 0\,.$$

Then, since according to our initial assumptions the derivatives of Φ are continuous in $\overline{\Omega}$ – i.e. also on $\partial\Omega$ – there exists on the boundary a neighbourhood $\delta\Omega \subset \partial\Omega$ of $x_0 \in \partial\Omega$ on which this derivative has the same sign. Consider now for u the harmonic measure defined by the boundary values:

$$u(\mathbf{x})|_{\mathbf{x}\in\partial\Omega} = \begin{cases} 1 & , \quad if\ \mathbf{x} \in \delta\Omega\,, \\ 0 & , \quad elsewhere\ on\ \partial\Omega. \end{cases} \tag{17}$$

In this way we find that if $\partial\Phi_{\mathfrak{k}}/\partial n$ is not identically zero on $\partial\Omega$, the integral (16) cannot be zero. As a consequence $\partial\Phi_{\mathfrak{k}}(\mathbf{x})/\partial n$ has to vanish identically on $\partial\Omega$ which finishes the proof of the theorem. ■

Let us write $Y(\mathbf{x})$ as $Y = Y_0 + Y_{\mathfrak{k}}$ where Y_0 belongs to the orthogonal complement of $Ker\mathcal{K}$ (it is the solution of minimal L^2 norm of equation (11)), and $Y_{\mathfrak{k}} \in Ker\mathcal{K}$. Similarly write

$$\Phi = \Phi_0 + \Phi_{\mathfrak{k}}\,, \quad \text{with} \quad \Phi(\mathbf{x}) = \chi_D(\mathbf{x}) + \int_{\Omega} d^n\mathbf{y}\, \mathcal{G}_D(\mathbf{x},\mathbf{y})\, Y_0(\mathbf{y}) \quad \text{and} \quad \Phi_{\mathfrak{k}}(\mathbf{x}) = \int_{\Omega} d^n\mathbf{y}\, \mathcal{G}_D(\mathbf{x},\mathbf{y}) Y_{\mathfrak{k}}(\mathbf{y})\,.$$

From the vanishing of the divergence of the current density $\sigma(\mathbf{x})\nabla\Phi(\mathbf{x})$ – i.e. from the eq.(1) – we obtain the identity

$$\int_{\Omega} d^2\mathbf{x}\, \sigma(\mathbf{x})\, \nabla\Phi(\mathbf{x}) \cdot \nabla\Phi(\mathbf{x}) = \int_{\partial\Omega} d\mathbf{x}\, \sigma(\mathbf{x})\, \Phi_0(\mathbf{x})\, \frac{\partial \Phi_0(\mathbf{x})}{\partial n}\,, \tag{18}$$

often called the energy functional. It is interesting to notice that according to our previous theorem, only $\Phi_0(\mathbf{x})$ appears in the right hand side of (18). It is clear that if the potential Φ_0 were zero, $\nabla\Phi(\mathbf{x})$ would vanish identically since the left hand side of (18) has the property of a norm. Consequently, $\Phi(\mathbf{x})$ would be a constant, namely equal to zero since its boundary values are zero in this case.

5. The regularized integral equation

Equation (11) is a Fredholm equation of the first kind which is notoriously ill-posed and whose solution is unstable with respect to the data measured on the boundary $\partial\Omega$. We shall regularize it by using a model function $Y_{mod}(\mathbf{x})$ as a first estimate of the solution. This function can be either an experimental or a theoretical one. The latter can be obtained starting with a model conductivity function σ_{mod} and solving equation (5), or better its first iterate (10), to find $Y_{mod}(\mathbf{x})$. The use of this regularizing function leads to a Fredholm equation of the second kind which can be solved explicitly and which is stable with respect to the errors in the input. Moreover the null space of $\mathcal{K}$ plays no particular role in the solution of Fredholm equation of the second kind.

Since the input data and hence the function $\chi(\mathbf{x})$ always contain errors we do not require that the left hand side of equation (11) is zero, but ask instead that its L^2 norm is small and that the solution Y_{reg} is close to a model function Y_{mod}

$$\|\chi(\mathbf{x}) \; - \; \mathbf{K}[Y_{reg}](\mathbf{x}) \, \|_{L^2} \to min, \quad \text{subject to} \quad \|\mathrm{Y_{reg}}(\mathbf{x}) \; - \; \mathrm{Y_{mod}}(\mathbf{x})\|_{\mathrm{L}^2} \leq \delta \,, \tag{19}$$

where $\mathbf{K}[Y_{reg}](\mathbf{x}) \; \equiv \; \int d^n\mathbf{y} \, \mathcal{K}(\mathbf{x},\mathbf{y}) \, Y_{reg}(\mathbf{y})$. In practical terms this means that we abandon the initial equation (11) and consider instead the Lagrange multiplier formulation related to the Tikhonov regularization [16], namely we seek the unrestricted minimum of the functional

$$\|\chi(\mathbf{x}) \; - \mathbf{K}[Y_{reg}](\mathbf{x})\|^2_{L^2} \; + \; \mu \, \left(\|Y_{reg}(\mathbf{x}) \; - \; Y_{mod}(\mathbf{x})\|^2_{L^2} \; - \; \delta^2\right) \; \to \; min$$

where μ is a Lagrange multiplier.

This yields the following regularized integral equation:

$$Y_{reg}(\mathbf{x}) \; = \; Y_{mod}(\mathbf{x}) \; + \; \lambda \int_\Omega d^n\mathbf{y} \, \mathcal{K}(\mathbf{x},\mathbf{y}) \, \chi(\mathbf{y}) \; - \; \lambda \int_\Omega d^n\mathbf{y} \, \mathcal{K}_2(\mathbf{x},\mathbf{y}) \, Y_{reg}(\mathbf{y}) \tag{20}$$

where $\mathcal{K}_2(\mathbf{x},\mathbf{y}) \; \equiv \; \int_\Omega d^n\mathbf{z} \, \mathcal{K}(\mathbf{x},\mathbf{z}) \, \mathcal{K}(\mathbf{z},\mathbf{y})$ and $\lambda \equiv 1/\mu$.

If the eigenfunctions $\{u_k\}$ of $\mathcal{K}(\mathbf{x},\mathbf{y})$ are known, we can expand the iterated kernel $\mathcal{K}_2$ as:

$$\mathcal{K}_2(\mathbf{x},\mathbf{y}) \; = \; \sum_{k=1}^{\infty} \frac{u_k(\mathbf{x}) \, u_k(\mathbf{y})}{\lambda_k^2} \, .$$

Moreover, if also the kernel $\mathcal{K}(\mathbf{x},\mathbf{y})$ can be expressed as

$$\mathcal{K}(\mathbf{x},\mathbf{y}) \; = \; \sum_{k=1}^{\infty} \frac{u_k(\mathbf{x}) \, u_k(\mathbf{y})}{\lambda_k} \, ,$$

this gives us a straightforward way to compute the eigenfunctions and eigenvalues. This is illustrated in the two dimensional example discussed in Section 6.

Projecting Eq.(20) onto u_k we obtain

$$Y_{reg,k} \; = \; Y_{mod,k} \; + \; \frac{\lambda}{\lambda_k} \chi_k \; - \; \frac{\lambda}{\lambda_k^2} Y_{reg,k}$$

where

$$Y_{reg,k} = \int_\Omega d^n\mathbf{x} \, Y_{reg}(\mathbf{x}) u_k(\mathbf{x}) \; , \; Y_{mod,k} = \int_\Omega d^n\mathbf{x} \, Y_{mod}(\mathbf{x}) u_k(\mathbf{x}) \; , \; \chi_k = \int_\Omega d^n\mathbf{x} \, \chi(\mathbf{x}) u_k(\mathbf{x}) \, .$$

This gives the following explicit expression for the solution $Y_{reg}(\mathbf{x})$:

$$Y_{reg}(\mathbf{x}) \; = \; Y_{mod}(\mathbf{x}) \; + \; \lambda \int_\Omega d\mathbf{y}^n \, \mathcal{K}(\mathbf{x},\mathbf{y}) \, \chi(\mathbf{y}) \; - \; \lambda \sum_{k=1}^{\infty} \frac{Y_{mod,k} \; + \; \dfrac{\lambda}{\lambda_k} \chi_k}{\lambda \; + \; \lambda_k^2} \, u_k(\mathbf{x}) \, . \tag{21}$$

Once $Y_{reg}(\mathbf{x})$ is known, the potential $\Phi_{reg}(\mathbf{x})$ can be obtained by means of the integral

$$\Phi_{reg}(\mathbf{x}) \; = \; \chi_D(\mathbf{x}) \; + \; \int_\Omega d^n\mathbf{y} \, \mathcal{G}_D(\mathbf{x},\mathbf{y}) \, Y_{reg}(\mathbf{y}) \, . \tag{22}$$

Finally, to determine the regularized $\sigma_{reg}(\mathbf{x})$, we can use the method of characteristics to solve the first order partial differential equation:

$$\nabla\widetilde{\sigma}_{reg}(\mathbf{x})\cdot\nabla\Phi_{reg}(\mathbf{x})-Y_{reg}(\mathbf{x})=0\,,\;\text{ for }\;\mathbf{x}\in\Omega \tag{23}$$

subject to the known boundary values $\widetilde{\sigma}(\mathbf{x})|_{\mathbf{x}\in\partial\Omega}$.

6. The two dimensional case: the unit disk

It is well known that any two dimensional simply connected domain can be mapped conformally onto the unit disk, where the relevant Green's functions are:

$$\mathcal{G}_D(r,\theta;\rho,\vartheta)\;=\;-\frac{1}{4\pi}\log\frac{r^2+\rho^2-2r\rho\cos(\theta-\vartheta)}{1+r^2\rho^2-2r\rho\cos(\theta-\vartheta)} \tag{24}$$

$$\mathcal{G}_N(r,\theta;\rho,\vartheta)\;=\;-\frac{1}{4\pi}\log\Big((r^2+\rho^2-2r\rho\cos(\theta-\vartheta))\cdot(1+r^2\rho^2-2r\rho\cos(\theta-\vartheta))\Big) \tag{25}$$

so that

$$\mathcal{K}(r,\theta;\rho,\vartheta)=\frac{1}{2\pi}\log\left(1+r^2\rho^2-2r\rho\cos(\theta-\vartheta)\right). \tag{26}$$

Hence the nonregularized integral equation (11) becomes:

$$0\;=\;\chi(r,\theta)\;-\;\int\limits_0^1\rho d\rho\int\limits_0^{2\pi}d\vartheta\,\frac{1}{2\pi}\log\left(1+r^2\rho^2-2r\rho\cos(\theta-\vartheta)\right)Y(\rho,\vartheta)\,. \tag{27}$$

where the kernel can be expanded as a Taylor series

$$\mathcal{K}(r,\theta;\rho,\vartheta)\;=\;-\sum_{k=1}^{\infty}\frac{r^k\rho^k\cos\,k(\theta-\vartheta)}{\pi k}.$$

This series expansion enables us to find the eigenfunctions and eigenvalues of $\mathcal{K}$. Indeed, the functions $u_k^1(r,\theta)\;=\;\sqrt{(2k+2)/\pi}\;r^k\cos\,k\theta$ and $u_k^2(r,\theta)\;=\;\sqrt{(2k+2)/\pi}\;r^k\sin\,k\theta$ are orthonormal on the unit disk, so that $\mathcal{K}$ can be rewritten as

$$\mathcal{K}(r,\theta;\rho,\vartheta)\;=\;\sum_{k=1}^{\infty}\sum_{j=1}^{2}\frac{u_k^j(r,\theta)\,u_k^j(\rho,\vartheta)}{\lambda_k}\,,$$

with $\lambda_k=-2k(k+1)$.

We can also compute explicitly the functions in the null space of $\mathcal{K}$. We begin with a function $Y_{\mathfrak{k}}$ of the form

$$Y_{\mathfrak{k}}(r,\theta)=\Re_k(r)\cos(k\theta)\,,\;\;k=1,2,3,\ldots\,.$$

which, in order to be in the null space must satisfy

$$\int_0^1\rho d\rho\int_0^{2\pi}d\vartheta\mathcal{K}(r,\theta;\rho,\vartheta)Y_{\mathfrak{k}}(\rho,\vartheta)=-\frac{\pi}{2k(k+1)}\int_0^1 d\rho\;\rho^{k+1}\Re_k(\rho)=0\,.$$

For this it is sufficient that

$$\Re_k(r,n)=G_n(k+2,k+2,r)\,,\text{for }\,n\geq1\,,$$

where $G_n(k+2,k+2,r)$ are the Jacobi polynomials of order n [17].
If we consider now

$$Y_{\mathfrak{k}}(r,\theta) = \Re_k(r)\sin(k\theta)\,, \quad k=1,2,3,\ldots\,.$$

we find a similar result. Hence the general form of a function in the null space of $\mathcal{K}$ is

$$Y_{\mathfrak{k}}(r,\theta) = \sum_k \sum_n G_n(k+2,k+2,r)\Big(A_{k,n}\cos(k\theta) + B_{k,n}\sin(k\theta)\Big)\,, \quad k=1,2,3,\ldots$$

where $A_{k,n}$ and $B_{k,n}$ are constants.

From expansion (21) we obtain the following explicit expression for the solution of the regularized equation

$$Y_{reg}(r,\theta) \;=\; Y_{mod}(r,\theta) + \lambda\int_0^1 \rho d\rho \int_0^{2\pi} d\vartheta\, \mathcal{K}(r,\theta;\rho,\vartheta)\,\chi(\rho,\vartheta) - \lambda \sum_{k=1}^{\infty}\sum_{j=1}^{2} \frac{Y^j_{mod,k} - \dfrac{\lambda}{2k(k+1)}\,\chi^j_k}{\lambda \,+\, 4k^2(k+1)^2}\, u^j_k(r,\theta). \tag{28}$$

To find σ_{reg} we proceed in the way discussed at the end of Section 5.

7. A Numerical Example

We have applied our algorithm to reconstruct the following conductivity distribution:

$$\sigma_{exact}(x,y) = \frac{1}{((x-0.3)^2+y^2)^{\frac{1}{2}} - 0.5)^2 + 1} + \frac{1}{((x-0.3)^2+y^2)^{\frac{1}{2}} + 0.5)^2 + 1}\,, \quad x^2+y^2 \le 1\,, \tag{29}$$

which reaches its maximum at $(x,y) = (0.3,0)$. We shall consider that the input current is

$$\sigma(1,\theta)\frac{\partial\Phi}{\partial n}\bigg|_{\partial\Omega}(\theta) = \cos(m\theta)\,, \quad m=1,2,3,\ldots\,.$$

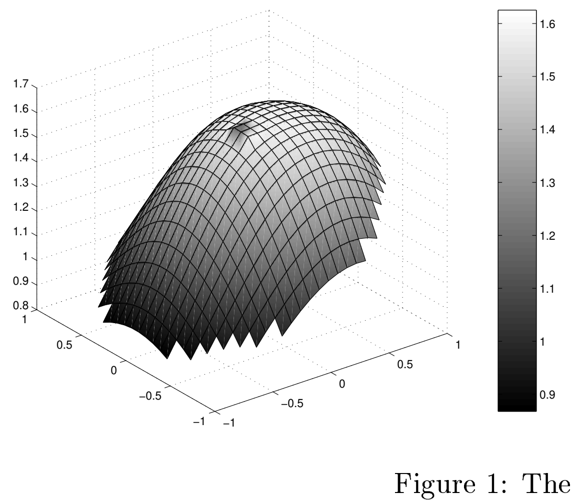

Figure 1: The exact conductivity.

To simulate the 'measured' values of the potential on the boundary we have to solve first the direct problem. With these data we can compute then the input function χ of our integral equation

$$\chi(r,\theta) = \int_0^{2\pi} d\vartheta\,\left\{\mathcal{G}_N(r,\theta;1,\vartheta)\,\frac{\partial\Phi}{\partial n}\bigg|_{\partial\Omega}(\vartheta) + \left[\frac{1}{2\pi} + \frac{\partial\mathcal{G}_D}{\partial n}(r,\theta;1,\vartheta)\right]\Phi(1,\vartheta)\right\}\,.$$

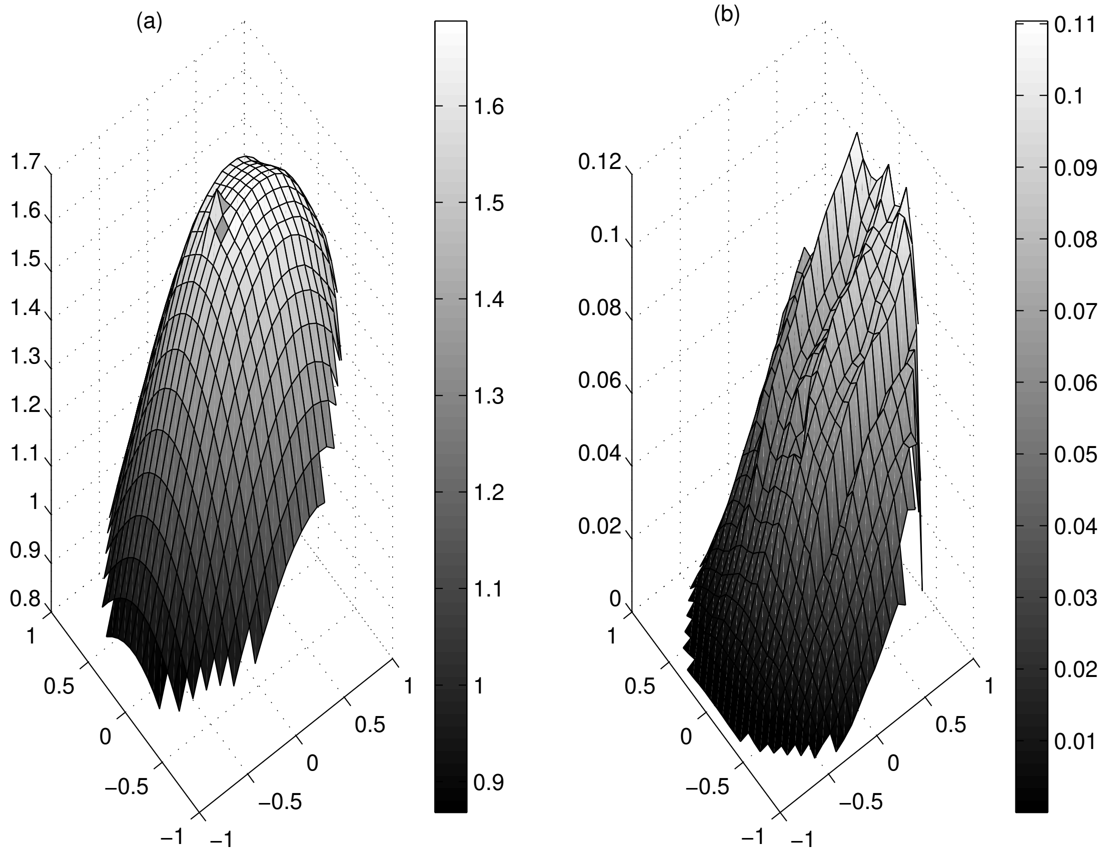

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Figure 2: The reconstructed conductivity
for data with random 2% errors
(a) - the reconstructed conductivity $\sigma(x,y)$
(b) - the absolute errors $\mid \sigma(x,y) - \sigma_{exact}(x,y) \mid$

By solving the forward problem as shown in Section 3, we find the model potential Φ_{mod} and hence also the model functions $Y_{mod}(x,y) = \nabla\widetilde{\sigma}_{mod}(x,y)\cdot\nabla\Phi_{mod}(x,y)$, where $\widetilde{\sigma}_{mod} = \ln(\sigma_{mod})$. The integral which appears in (28) can be computed numerically:

$$\int_0^1 \rho d\rho \int_0^{2\pi} d\vartheta \mathcal{K}(r,\theta;\rho,\vartheta)\chi(\rho,\vartheta) = \sum_{l=1}^{172} w_l \mathcal{K}(r,\theta;r_l,\theta_l)\chi(r_l,\theta_l)\,,$$

where $\{w_l; r_l, \theta_l\}$ is a set of quadrature weights and points for the unit disk [18]. Since, the infinite series which appear in equation (28) converges rapidly it was sufficient to consider only fifty terms.

There are many ways of determining the value of the parameter λ – e.g. see [16], [19] – for instance using the condition (19) for some given value of the precision δ. Once Y_{reg} is known, the calculation of the potential inside the unit disk is straightforward:

$$\Phi(r,\theta) = \chi_D(r,\theta) + \sum_{l=1}^{172} w_l \mathcal{G}_D(r,\theta;r_l,\theta_l) Y_{reg}(r_l,\theta_l)\,,$$

where

$$\chi_D(r,\theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{\Phi(1,\vartheta)}{1+r^2-2r\cos(\theta-\vartheta)} d\vartheta\,.$$

The final step is to solve our first order partial differential equation (23). The method of characteristics transforms this equation into an equivalent system of ordinary differential equations which can be written in cartesian coordinates as:

$$\frac{dx}{ds} = \frac{d\Phi}{dx}(x(s),y(s))\,,\ \ \frac{dy}{ds} = \frac{d\Phi}{dy}(x(s),y(s))\,,\ \ \frac{d\widetilde{\sigma}}{ds} = Y_{reg}(x(s),y(s))\,. \tag{31}$$

We have used sixty characteristic curves, the initial conditions for the i-th characteristic being:

$$x_0^i = \cos\theta_0^i\,,\ \ y_0^i = \sin\theta_0^i\,,\ \ \theta_0^i = 2\pi/i\,,\ \ for\ i = 1\ldots 60\,, \tag{32}$$

$$\widetilde{\sigma}_0^i = \ln\left(\frac{1}{(1.09 - 0.6\cos\theta_0^i)^2)^{\frac{1}{2}} - 0.5)^2 + 1} + \frac{1}{(1.09 - 0.6\cos\theta_0^i)^{\frac{1}{2}} + 0.5)^2 + 1}\right)\,.$$

We present in figure (2) below the results obtained for $m = 1$, for data with 2% random errors.

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