Introduction to nuclear and particle physics
- Lecture notes -

These lecture notes are based on the course given to young scientists (most of them being PhD students), working on temporary positions at IFIN-HH, Bucharest. They cover most of the necessary background concerning nuclear and particle physics.
The material is organized in five Sections:

I. Nuclear structure (D.S. Delion)
II. Observables in nuclear physics (A. Negret)
III. Dosimetry (A. Stochioiu)
IV. Experimental nuclear physics (F. Constantin)
V. Particle physics (G. Stoicea).

The lectures are given by senior researchers of the institute (mentioned in paran-thesis), sharing in this way their knowledge and expertise to young scientists.
In the competition for a permanent position in our institute the students should pass an examination proving the level of knowledge of these lecture notes.

Bucharest, September 2013
D.S. Delion, A. Negret, A. Stochioiu, F. Constantin, G. Stoicea

I. Nuclear structure

This Section gives the theoretical background on quantum physics concerning nuclear phenomena at low energies. A short overview of quantum mechanics with applications in nuclear physics is also given. The material is organized in four Chapters:

A. Basics of quantum mechanics
B. Phenomenological nuclear models
C. Microscopic nuclear models
D. Decay processes

In order to have a better understanding, we recommend a basic literature which was considered as a source of these lecture notes.

Bucharest, September 2013 D.S. Delion

Books

Quantum mechanics

S. Titeica, “Mecanica cuantica”
P.A.M. Dirac, “The principles of quantum mechanics”
L.I.Shiff, “Quantum mechanics”
E. Merzbacher, “Quantum mechanics”
N. Zettili, “Quantum mechanics. Concepts and Applications”

Nuclear physics

P. Ring, P. Schuck, “Nuclear many body problem”
K. Heyde, “Basic concepts and ideas in nuclear physics”
W. Greiner, J.A. Maruhn, “Nuclear models”
K.S. Krane, “Introductory nuclear physics”
S. Cadogan, “Nuclear physics”

A. Basics of quantum mechanics

QUANTUM OBSERVABLES

A.14. Addition of angular momenta
A.15. Spin-orbit coupling
A.16. Pairing coupling
A.17. Spin of nuclei
A.18. Electric transitions
A.19. Magnetic dipole momentum

QUANTUM DYNAMICS

A.20. Schrodinger equation
A.21. One dimensional harmonic oscillator
A.22. Dinuclear systems
A.23. Deuteron
A.24. Nuclear two body potential

A.01. Introduction
Fundamental interactions

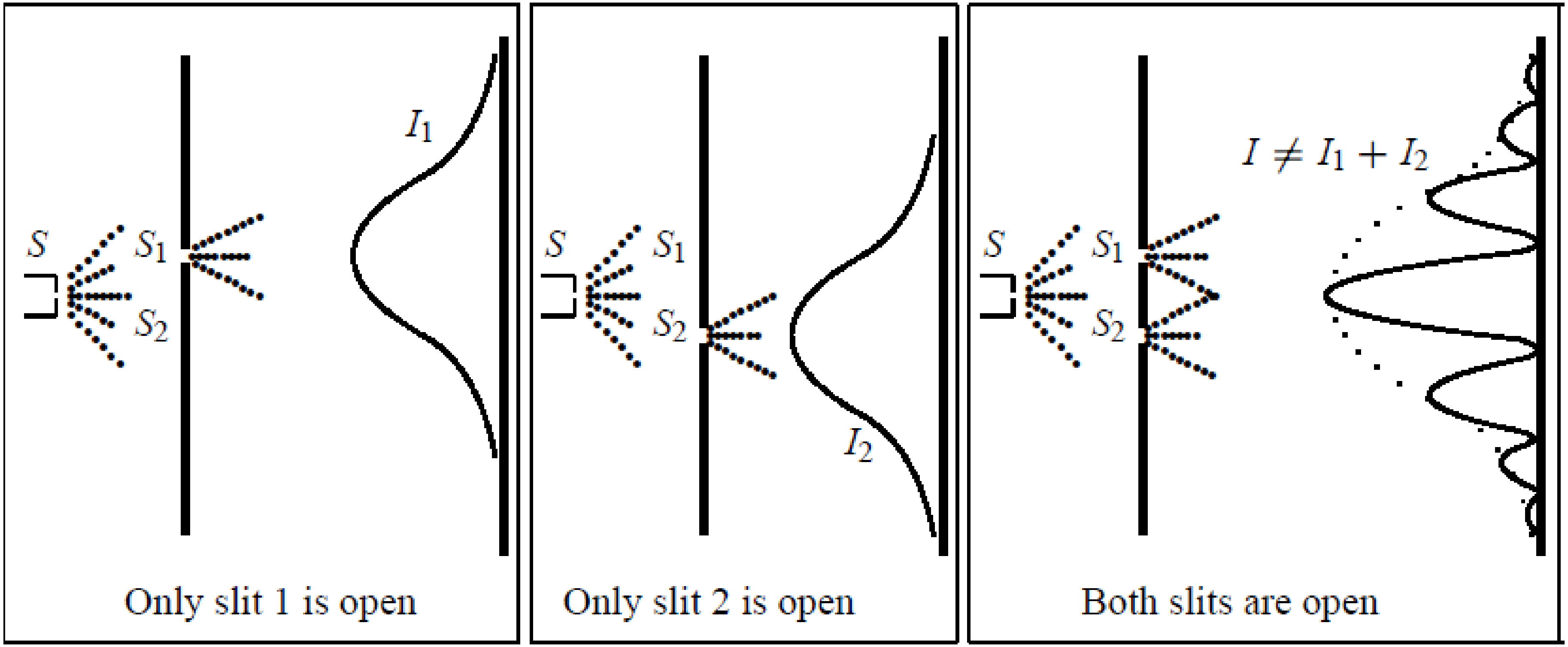
have the following strengths
(by black: 1 = the strongest
by red: 1 = the weakest):

1) Gravitational interaction:	10^{-38}	(1)
2) Weak interaction (beta decay):	10^{-13}	(10^{25})
3) Electromagnetic interaction:	10^{-2}	(10^{36})
4) Strong interaction (nuclear):	1	(10^{38})

Standard model
describes in an unified way 2+3+4
within a relativistic quantum formalism.

We will describe separately 2, 3 & 4 within the
non-relativistic quantum limit
valid for energies much lower than the nucleonic mass (1 GeV)

Interference picture of electrons passing through two slits leads to the dual particle-wave nature of particles described by the quantum mechanics



The double-slit experiment: S is a source of *electrons*, I_1 and I_2 are the intensities recorded on the screen when only S_1 is open, and then when only S_2 is open, respectively. When both slits are open, the total intensity is equal to the sum of I_1, I_2 and an *oscillating* term.

A.02. Quantization principles

are based on the analogy
between the optics and mechanics

Wave optics, λ~d → **Quantum mechanics**

Geometrical optics, λ<<d
propagation along rays → **Classical mechanics**
propagation along trajectories

where d is the slit dimension and λ the wave length

Old quantum mechanics (<1924)

quantizes the action according to the condition
describing stationary waves along a classical trajectory:

$$\oint k dr = \oint \frac{2\pi}{\lambda} dr = 2\pi n \Rightarrow \oint p dr = 2\pi\hbar n = hn$$

Thus, the "elementary cel" in the phase space is $2\pi\hbar$ / degree of freedom.

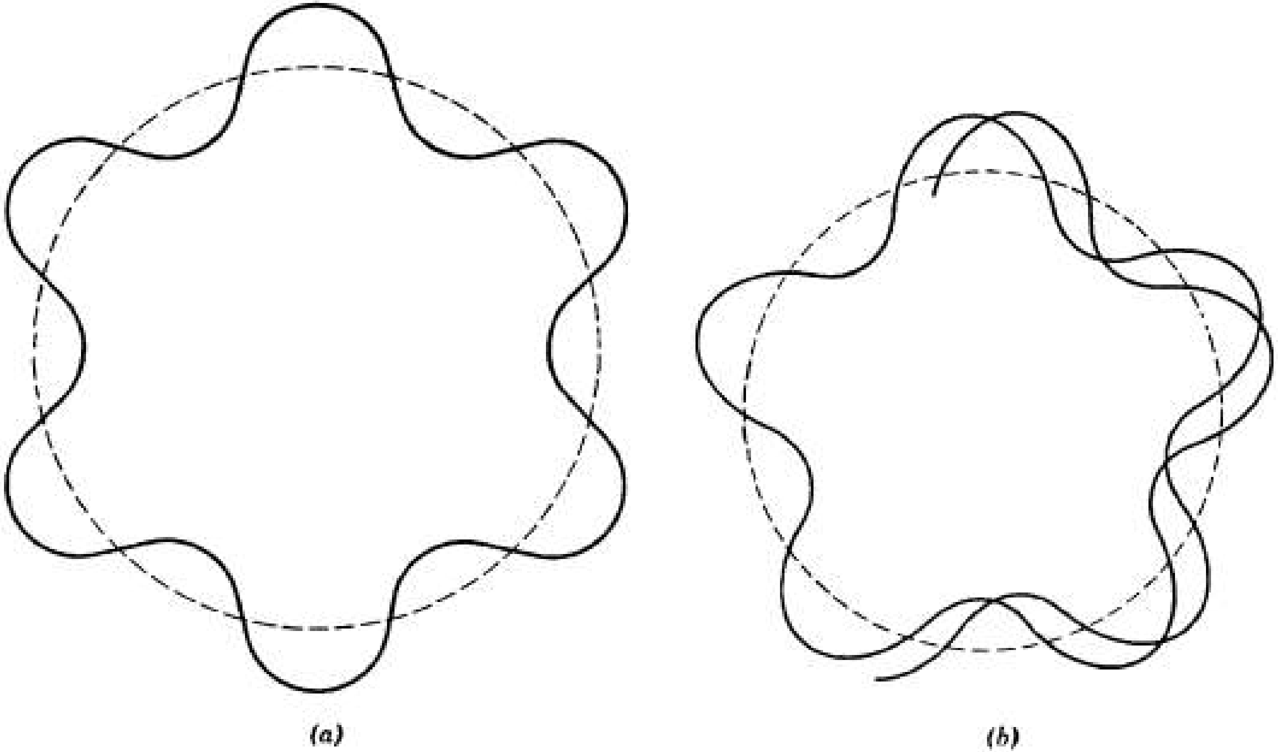


Figure 1.1. (*a*) Constructive interference of de Broglie waves in an atom distinguishes the allowed stable Bohr orbits. (*b*) Destructive interference of de Broglie waves in an atom disallows any orbit which fails to satisfy the quantum conditions.

Modern quantum mechanics (>1924)

is based on three principles:

a. Equivalence principle (Louis de Broglie)

Wave number: $k=2\pi/\lambda$ → Momentum: $\mathbf{p}=\hbar\mathbf{k}$

Circular frequency: $\omega=2\pi/T$ → Energy: $E=\hbar\omega$

Reduced Plank constant

$\hbar=h/2\pi\approx10^{-34}$ J.s

has the dimension of the action

[h]~ **p**.**r** (momentum.space) or E.t (energy.time)

b. Statistical principle (Max Born)

The state of a particle is described by a wave function which has a probabilistic meaning: the square of its amplitude is the probability to find the particle in this state.

c. Superposition principle

The sum of two wave functions describing physical states is also a wave function describing a physical state.

First example:
wave function of the free motion is a plane wave

$$\psi(t,r) = Ae^{i(\mathbf{k}.\mathbf{r}-\omega t)} \equiv Ae^{i(k_x x+k_y y+k_z z-\omega t)}$$

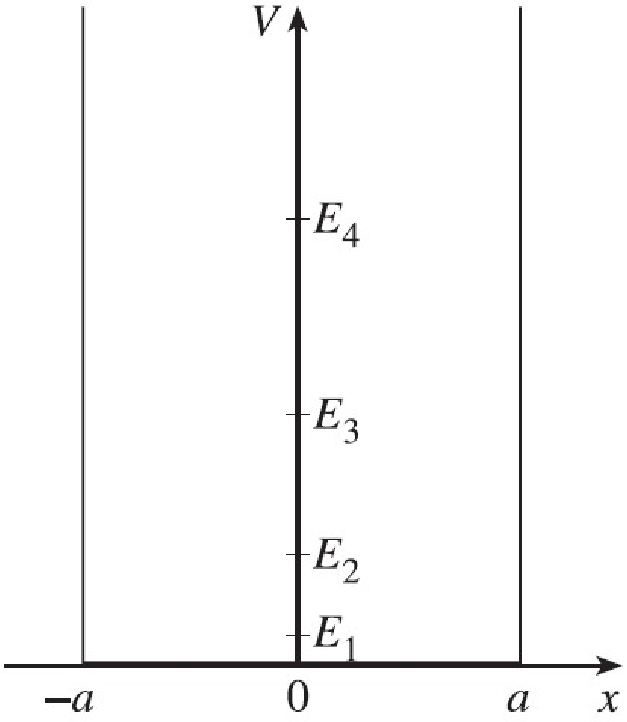
$$\to Ae^{\frac{i}{\hbar}(\mathbf{p}.\mathbf{r}-Et)} \equiv Ae^{\frac{i}{\hbar}(p_x x+p_y y+p_z z-Et)}$$

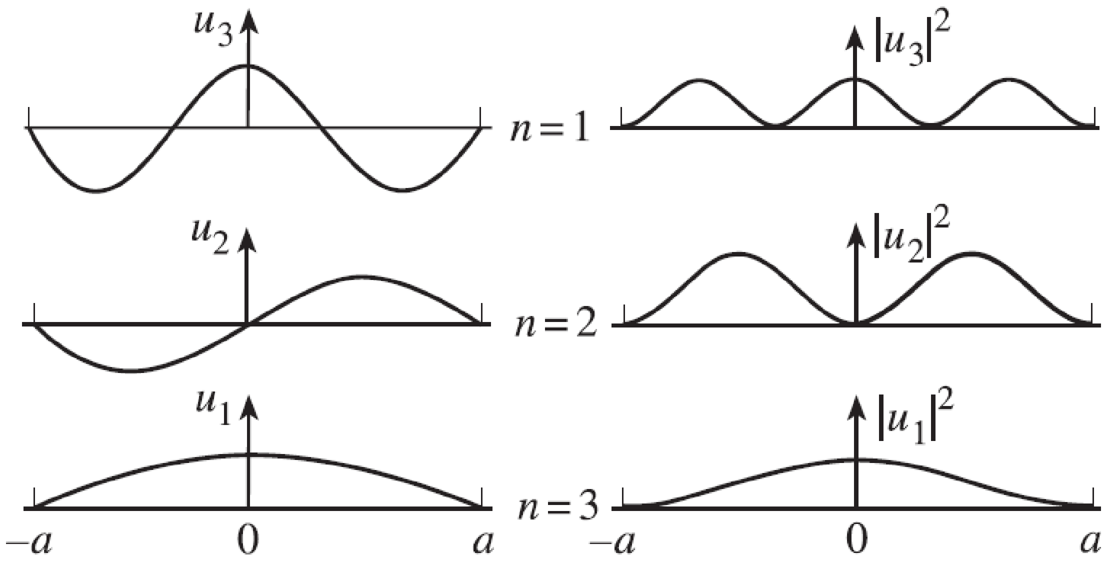
The probability to have:
a given energy *E* for any time *t* and
a given momentum **p** at any point in space **r**
is a constant $P=|A|^2$
The values of the energy and momentum are not restricted and
the probability cannot be normalized to unity because
the wave function is extended at infinity.

Second example:

wave function of a particle in a potential well

has as an analog the stationary waves vanishing beyond the walls, with a restricted spectrum of frequencies (energies)





For the bound motion the probability can be normalized to unity because the wave function vanishes at large distances.

As a consequence, the energy spectrum becomes discrete

A.03. Wave function. Dirac bra-ket notation

A “ket” wave function can be represented in the configuration space **x** as a column set of states

$$|\psi\rangle \rightarrow \begin{bmatrix} \langle x_1 | \psi \rangle \\ \dots\dots\dots \\ \langle x_n | \psi \rangle \end{bmatrix} \equiv \begin{bmatrix} \psi(x_1) \\ \dots\dots\dots \\ \psi(x_n) \end{bmatrix}$$

A “bra” wave function can be represented in the configurations space **x** as a row set of hermitian transposed states

$$\langle \phi | \rightarrow \left[\langle \phi | x_1 \rangle^*, \dots\dots, \langle \phi | x_n \rangle^* \right]$$
$$\equiv \left[\phi^*(x_1), \dots\dots, \phi^*(x_n) \right]$$

Their scalar product has a “bra-ket” form (from bracket):

$$\left[\langle \phi | x_1 \rangle^*, \dots\dots, \langle \phi | x_n \rangle^* \right] . \begin{bmatrix} \langle x_1 | \psi \rangle \\ \dots\dots\dots \\ \langle x_n | \psi \rangle \end{bmatrix} = \sum_{k=1}^{n} \langle \phi | x_k \rangle^* \langle x_k | \psi \rangle = \langle \phi | \psi \rangle$$

The generalization to the continuum case is obvious, replacing summation by integration

A.04. Observables

correspond to physical variables
and are defined by the expectation values of
the corresponding **quantum operators**
on the wave function

$$\Lambda = \int \psi^*(\mathbf{r})\hat{\mathbf{\Lambda}}\psi(\mathbf{r})d\mathbf{r} \equiv \left\langle \psi \,|\, \hat{\mathbf{\Lambda}} \,|\, \psi \right\rangle$$

with the normalisation condition
giving the total probability:

$$\int \psi^*(\mathbf{r})\psi(\mathbf{r})d\mathbf{r} \equiv \left\langle \psi \,|\, \psi \right\rangle = 1$$

A.05. Conjugate quantum operators

Operators entering as products in the action
are called conjugate operators

Examples of one dimensional conjugate operators:

Cartesian coordinate - Linear momentum

Angle - Angular momentum

Time - Energy

$$\hat{x} = x \qquad \hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

because:

$$\frac{\hbar}{i}\frac{\partial}{\partial x} e^{\frac{i}{\hbar} p_x x} = p_x$$

$$\hat{\varphi} = \varphi \qquad \hat{L}_\varphi = \frac{\hbar}{i}\frac{\partial}{\partial \varphi}$$

$$\hat{t} = t \qquad \hat{H}_x = \frac{\hat{p}_x{}^2}{2m} + V = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V$$

Generalization to three dimensions

$$\hat{\mathbf{r}} = \mathbf{r} \qquad \hat{\mathbf{p}} = \frac{\hbar}{i}\nabla \equiv \frac{\hbar}{i}\left(\mathbf{e}_{\mathbf{x}}\frac{\partial}{\partial x} + \mathbf{e}_{\mathbf{y}}\frac{\partial}{\partial y} + \mathbf{e}_{\mathbf{z}}\frac{\partial}{\partial z}\right)$$

$$\hat{\varphi} = \varphi \qquad \hat{\mathbf{L}} = \mathbf{r} \times \hat{\mathbf{p}}$$

$$\hat{t} = t \qquad \hat{\mathbf{H}} = \frac{\hat{\mathbf{p}}^2}{2m} + V = -\frac{\nabla^2}{2m} + V$$

The energy operator $\hat{\mathbf{H}}$ is called Hamiltonian

A.06. Uncertainty relations

Conjugate operators do not commute

Example: coordinate-momentum

$$\frac{\hbar}{i}\left[x\frac{\partial f}{\partial x}-\frac{\partial(fx)}{\partial x}\right]=-\frac{\hbar}{i}f$$

$$\Downarrow$$

$$[\mathbf{r},\hat{\mathbf{p}}]=i\hbar\hat{\mathbf{I}}$$

where $\hat{\mathbf{I}}$ is the unity operator

By using Schwartz inequality
one can show that:

$$\Delta A \Delta B \geq \left| \frac{1}{2} \left\langle [\hat{\mathbf{A}}, \hat{\mathbf{B}}] \right\rangle \right|$$

where we defined the uncertainty:

$$\Delta A = \sqrt{\left\langle \hat{\mathbf{A}}^2 \right\rangle - \left\langle \hat{\mathbf{A}} \right\rangle^2}$$

in terms of the difference between the operator
and its expectation value

$$\left\langle \hat{\mathbf{A}} \right\rangle \equiv \hat{\mathbf{A}} - \left\langle \Psi \,|\, \hat{\mathbf{A}} \,|\, \Psi \right\rangle$$

This implies

uncertainty (Heisenberg) relations

for coordinate-momentum

$$\Delta x \Delta \hat{p}_x \geq \frac{\hbar}{2}$$

$$\Delta y \Delta \hat{p}_y \geq \frac{\hbar}{2}$$

$$\Delta z \Delta \hat{p}_z \geq \frac{\hbar}{2}$$

Conclusions:

1) the observables corresponding to non-commuting variables cannot be simultaneously measured, as for example in the case of conjugate variables;

2) only the observables corresponding to commuting operators can be simultaneously measured.

For time-energy one has a similar relation

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Thus, a stationary state with a given energy
has an infinite life and
a state with a half life T has the energy width

$$\Gamma = 2\Delta E \approx \frac{\hbar}{T}$$

A.07. Eigenstates

One dimensional eigenstates of an operator
are defined by the following equation:

$$\hat{\mathbf{\Lambda}}\psi_n(x) = \lambda_n \psi_n(x)$$

$$n = 1, 2, \ldots, \infty$$

with given boundary conditions on some interval *[a,b]*

$$\psi_n(a) = \psi_n(b) = 0$$

The eigenvalues λ_n correspond to the measured values, because
the.expectation value on eigenstates gives the set observable eigenvalues

$$\int_a^b \psi_n^*(\mathbf{r})\hat{\mathbf{\Lambda}}\psi_n(\mathbf{r})d\mathbf{r} \equiv \left\langle \psi_n \,|\, \hat{\mathbf{\Lambda}} \,|\, \psi_n \right\rangle = \lambda_n \langle \psi_n \,|\, \psi_n \rangle = \lambda_n$$

$$\hat{\mathbf{\Lambda}}\psi_n(x) = \lambda_n\psi_n(x)$$

$$\hat{\mathbf{\Lambda}}\psi_m(x) = \lambda_m\psi_m(x)$$

By multiplying first equality to left with $\psi_m{}^*$,
the second one with $\psi_n{}^*$,
by integrating and substracting them

if the operator is symmetric
(or self-adjoint):

$$\int_a^b \psi_m^*(x)\hat{\mathbf{\Lambda}}\psi_n(x)dx \equiv \left\langle \psi_m \mid \hat{\mathbf{\Lambda}} \mid \psi_n \right\rangle =$$

$$\int_a^b \psi_n^*(x)\hat{\mathbf{\Lambda}}\psi_m(x)dx \equiv \left\langle \psi_n \mid \hat{\mathbf{\Lambda}}^+ \mid \psi_m \right\rangle$$

then the eigenstates corresponding to different
eigenvalues are orthonormal

$$\int_a^b \psi_n^*(x)\psi_m(x)dx \equiv \left\langle \psi_n \mid \psi_m \right\rangle = \delta_{nm}$$

A state can be expanded in terms of the eigenstates

$$\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

where the coefficients are given by

$$c_n = \int_a^b \psi_n^*(x)\psi(x)dx \equiv \langle \psi_n | \psi \rangle$$

Thus, the above expansion can be written in the bra-ket notation as follows:

$$|\psi\rangle = \sum_{n=1}^{\infty} |\psi_n\rangle\langle\psi_n|\psi\rangle$$

or, as a resolution of the unity operator:

$$\hat{\mathbf{I}} = \sum_{n=1}^{\infty} |\psi_n\rangle\langle\psi_n|$$

By multiplying to left with ψ^* and integrating the eigenvalue equation, it can be written in a matrix form as follows:

$$\sum_{m=1}^{\infty} c_m^* \int_a^b \psi_m^*(x)\hat{\Lambda}\psi_n(x)dx \equiv \sum_{m=1}^{\infty} \langle \psi_m | \Lambda | \psi_n \rangle c_m^* = \lambda_n c_n^*$$

The matrix

$$\Lambda_{mn} \equiv \int_a^b \psi_m^*(x)\hat{\Lambda}\psi_n(x)dx \equiv \left\langle \psi_m | \hat{\Lambda} | \psi_n \right\rangle$$

is called the matrix representation of the operator Λ in the basis ψ_n

The vector of coefficients c_n define the wave function ψ in the basis ψ_n

A.08. Quantum measurement

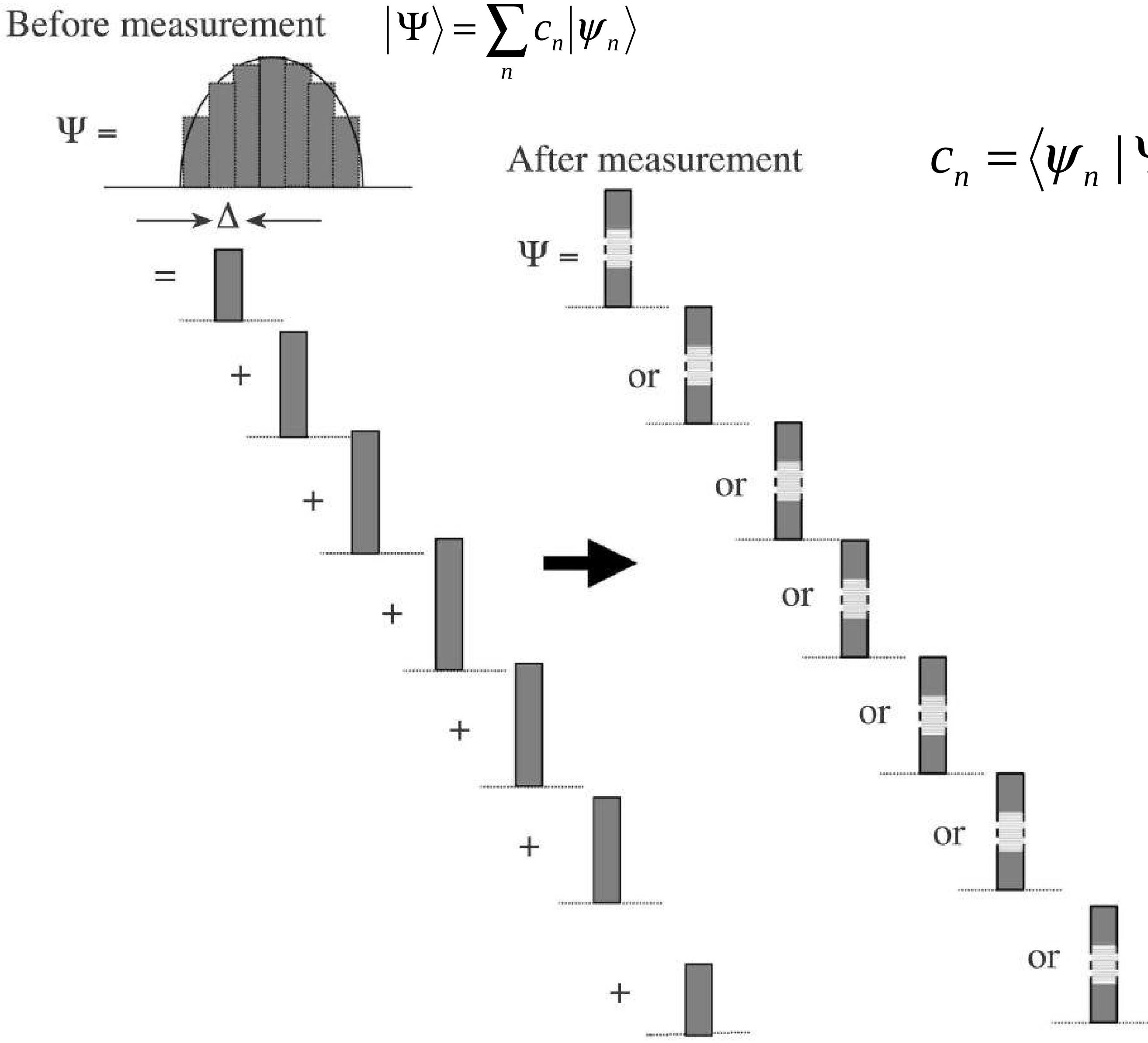
Due to the orthonormality of the basis states
the total probability is a sum of partial probabilities

$$\int_a^b \left|\psi(x)\right|^2 dx = 1 = \sum_{n=1}^{\infty} c_n^2$$

This is the reason why the coefficient c_n
is called the amplitude to find (or to measure)
the state ψ_n in the state ψ.

$$\left|\Psi\right\rangle = \sum_n c_n \left|\psi_n\right\rangle$$

$$c_n = \left\langle \psi_n \mid \Psi \right\rangle$$



A.09. One dimensional box [-*L/2* , *L/2*]

The stationary wave function
with periodic conditions at x = ±L/2 and
normalized in this interval is given by:

$$\psi_n(x) = \frac{1}{\sqrt{L}} e^{ik_n x}$$

where momenta
are quantized :

$$k_n = \frac{p_n}{\hbar} = n\Delta k$$

in terms of the fundamental
wave number:

$$\Delta k = \frac{2\pi}{L}$$

One obtains the following relation
in the limit of an infinite large box:

$$\psi(x) = \sum_{n=-\infty}^{\infty} c_n \frac{e^{ik_n x}}{\sqrt{L}} = \sum_{n=1}^{\infty} \int_{-L/2}^{L/2} \frac{e^{-ik_n y}}{\sqrt{L}} \psi(y) dy \frac{e^{ik_n x}}{\sqrt{L}} = \sum_{n=-\infty}^{\infty} \Delta k \int_{-L/2}^{L/2} \frac{e^{ik_n (x-y)}}{2\pi} \psi(y) dy$$

$$\xrightarrow[L\to\infty]{} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dk \frac{e^{ik(x-y)}}{2\pi} \psi(y) = \int_{-\infty}^{\infty} dy \delta(x-y) \psi(y) = \psi(x)$$

where delta-function is defined by the last equality

In this way, one gets the following representation of the delta function:

$$\int_{-\infty}^{\infty} e^{ik(x-y)} dk = 2\pi\delta(x-y)$$

The basis for the free single particle stationary motion
in three dimensions is given by the product of three plane waves:

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}.\mathbf{r}}$$

$$= \langle \mathbf{k} | \mathbf{r} \rangle = \langle \mathbf{r} | \mathbf{k} \rangle^{*}$$

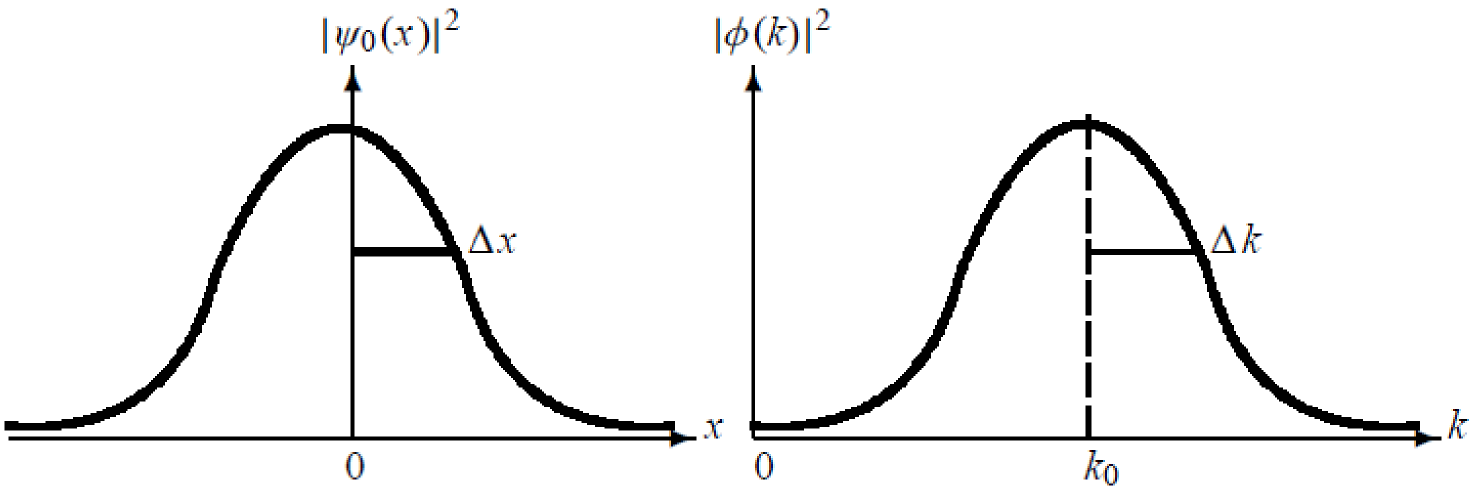
Thus, the wave function can be considered
1) depending on coordinate **r** for a given momentum **k** or
2) depending on momentum **k** for a given coordinate **r**.
These states are normalized to the delta function
in both coordinate and momentum space

$$\int \frac{e^{i\mathbf{k}.(\mathbf{r}-\mathbf{r}')}}{(2\pi)^3} d\mathbf{k} = \delta(\mathbf{r}-\mathbf{r}'); \_\_\_\_\_\_ \int \frac{e^{i(\mathbf{k}-\mathbf{k}').\mathbf{r}}}{(2\pi)^3} d\mathbf{r} = \delta(\mathbf{k}-\mathbf{k}')$$

Illustration of the uncertainty relation

for a stationary one dimensional Gaussian wave-packet

$$\psi_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_0(k) e^{-ikx} dx$$



Two localized wave packets: $\psi_0(x) = (2/\pi a^2)^{1/4} e^{-x^2/a^2} e^{ik_0 x}$

and $\phi(k) = (a^2/2\pi)^{1/4} e^{-a^2(k-k_0)^2/4}$

they peak at $x = 0$ and $k = k_0$, respectively, and vanish far away.

A.10. Angular momentum

Angular momentum operator is defined by:

$$\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} = \frac{\hbar}{i} \mathbf{r} \times \nabla$$

$$\hat{L}_x = y\hat{p}_z - z\hat{p}_y$$

$$\hat{L}_y = z\hat{p}_x - x\hat{p}_z$$

$$\hat{L}_z = x\hat{p}_y - y\hat{p}_z$$

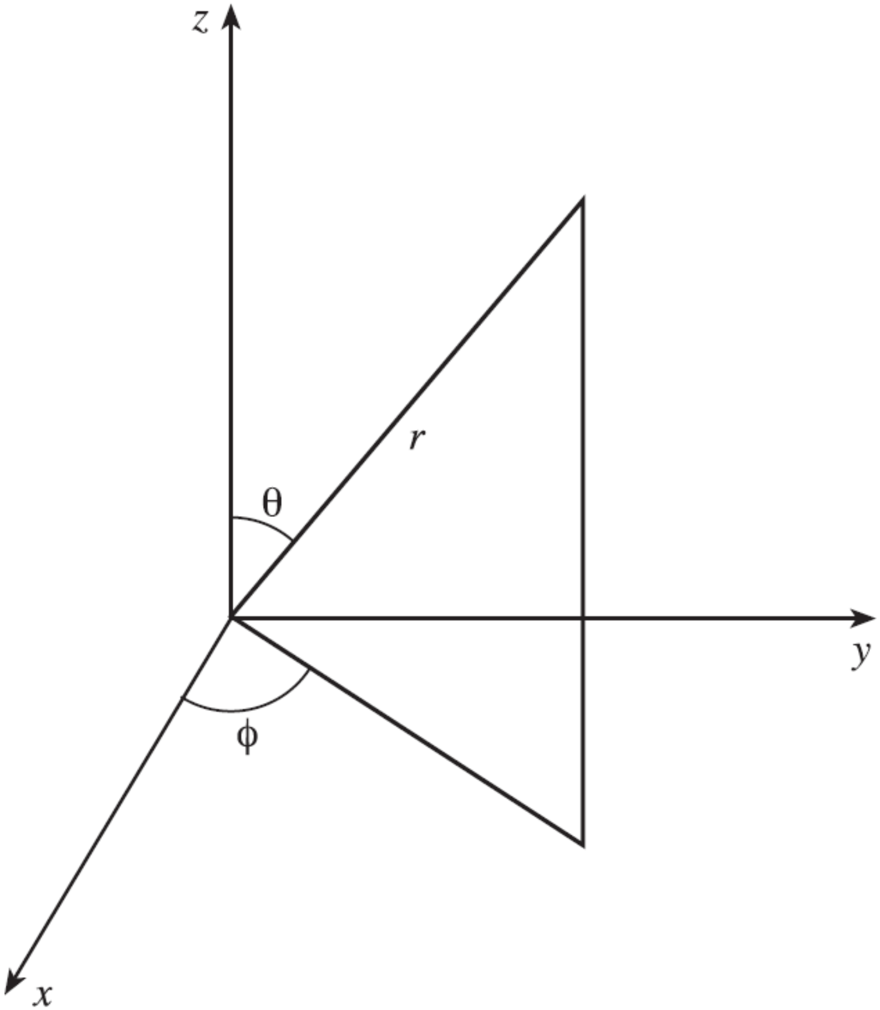
The components satisfy the commutation relations
and therefore they cannot be simultaneously measured

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z$$

$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x$$

$$\left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y$$

Angular momentum in spherical coordinates



$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

The angular momentum squared has the following form:

$$\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2 \left[\frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \left(\sin\vartheta \frac{\partial}{\partial\vartheta} \right) + \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial\varphi^2} \right]$$

It commutes with one of the components, for instance

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial\varphi}$$

Therefore they have the same eigenfunctions
called spherical functions:

$$\hat{\mathbf{L}}^2 Y_{lm}(\vartheta,\varphi) = \hbar^2 l(l+1) Y_{lm}(\vartheta,\varphi) \Leftrightarrow \hat{\mathbf{L}}^2 \left| lm \right\rangle = \hbar^2 l(l+1) \left| lm \right\rangle$$

$$\hat{L}_z Y_{lm}(\vartheta,\varphi) = \hbar m Y_{lm}(\vartheta,\varphi) \Leftrightarrow \hat{L}_z \left| lm \right\rangle = \hbar m \left| lm \right\rangle$$

depending on *l=0,1,2,…*, called angular momentum quantum number and
m=-l,-l+1,…-1,0,1,…l-1,l, (i.e. 2l+1) values) called magnetic quantum number

Spherical functions (spherical harmonics)

generalize trigonometric functions from circle to sphere

$$Y_{lm}(\vartheta,\varphi) \equiv \langle \vartheta,\varphi \mid lm \rangle$$

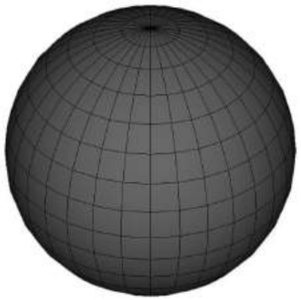
They are products between generalized Legendre polinomials in *cos θ* and trigonometric exponents in *φ*:

$$Y_{lm}(\vartheta,\varphi) = P_l^{(m)}(\cos\vartheta)e^{im\varphi}$$

Spherical harmonics are orthonormal on sphere:

$$\int_{-1}^{1} d\cos\vartheta \int_{0}^{2\pi} d\varphi Y_{lm}^{*}(\vartheta,\varphi) Y_{l'm'}(\vartheta,\varphi) \equiv \langle lm \mid l'm' \rangle = \delta_{ll'}\delta_{mm'}$$

Some spherical functions
in spherical and cartezian coordinates



0,0

$$Y_{00}(\vartheta,\varphi)=\frac{1}{\sqrt{4\pi}}$$

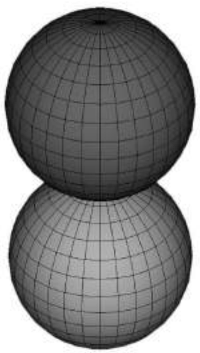
$$Y_{10}(\vartheta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\vartheta=\sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$$

$$Y_{1\pm1}(\vartheta,\varphi)=\mp\sqrt{\frac{3}{8\pi}}e^{\pm i\varphi}\sin\vartheta=\mp\sqrt{\frac{3}{8\pi}}\,\frac{x\pm iy}{r}$$

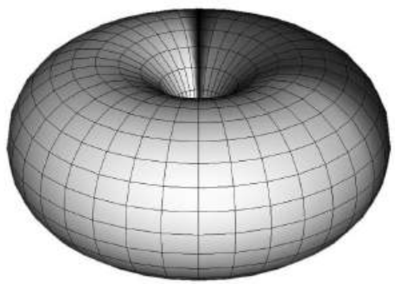
$$Y_{20}(\vartheta,\varphi)=\sqrt{\frac{5}{16\pi}}\left(3\cos^2\vartheta-1\right)=\sqrt{\frac{5}{16\pi}}\,\frac{3z^2-r^2}{r^2}$$

$$Y_{2\pm1}(\vartheta,\varphi)=\sqrt{\frac{15}{8\pi}}e^{\pm i\varphi}\sin\vartheta\cos\vartheta=\sqrt{\frac{15}{8\pi}}\,\frac{(x\pm iy)z}{r^2}$$

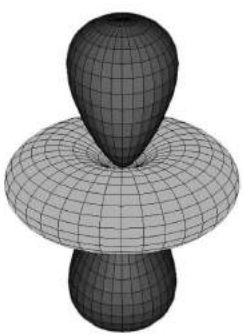
$$Y_{2\pm2}(\vartheta,\varphi)=\sqrt{\frac{15}{32\pi}}e^{\pm2i\varphi}\sin^2\vartheta=\sqrt{\frac{15}{32\pi}}\,\frac{x^2-y^2\pm2ixy}{r^2}$$



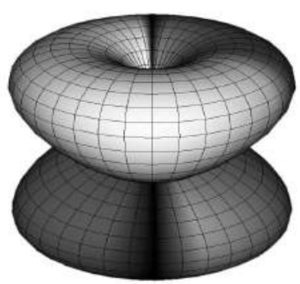
1,0



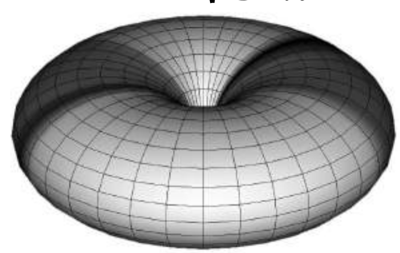
1,±1



2,0

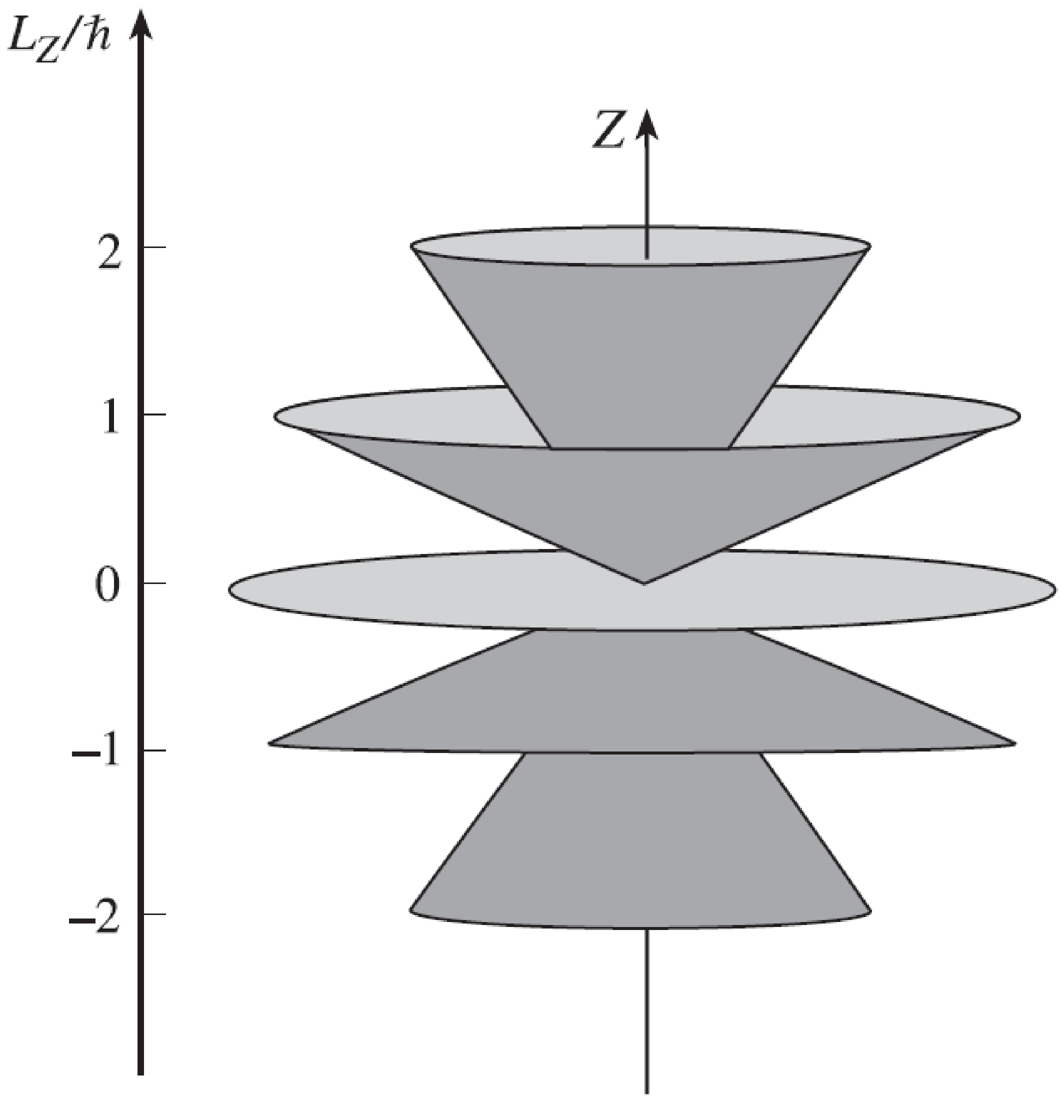


2,±1



2,±2

The possible projections of the angular momentum l=2 in space are quantized by 2l+1=5 values, according to the magnetic quantum number m=-2,-1,0,1,2



A.11. Parity

The quantum state has a parity, given by the reflection
with respect to the origin of the coordinate system.
The parity operator has two eigenvalues: ±1

$$\hat{\mathbf{P}}\psi(r) =$$

$$\psi(-\mathbf{r}) = +\psi(\mathbf{r})$$ even states

$$\psi(-\mathbf{r}) = -\psi(\mathbf{r})$$ odd states

In spherical coordinates $\mathbf{r} \to -\mathbf{r} \Rightarrow \vartheta \to \pi - \vartheta; \varphi \to \pi + \varphi$

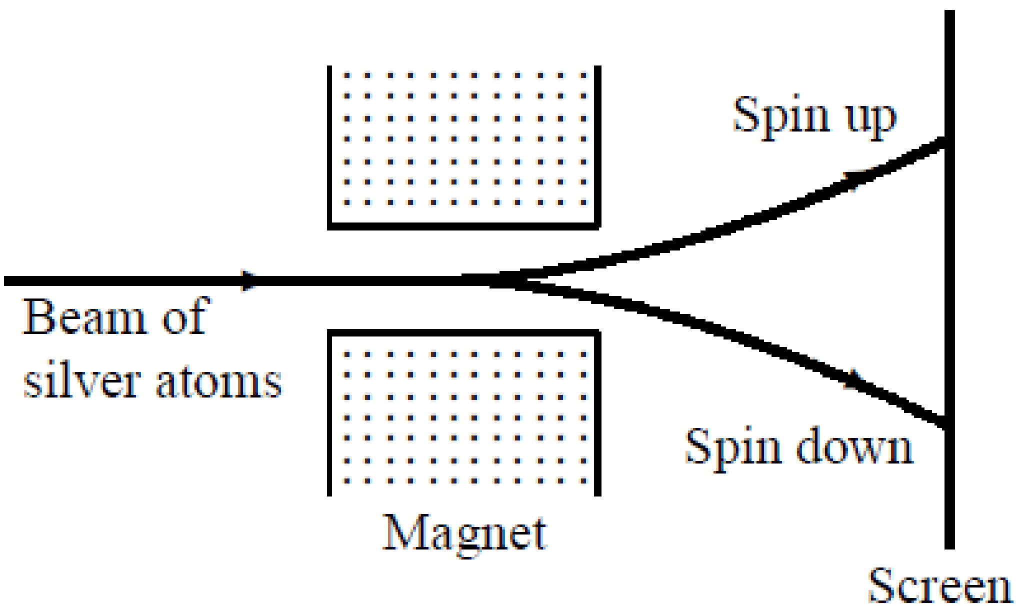
A state with the angular momentum *l*
has positive parity for even *l*
and negative parity for odd *l*

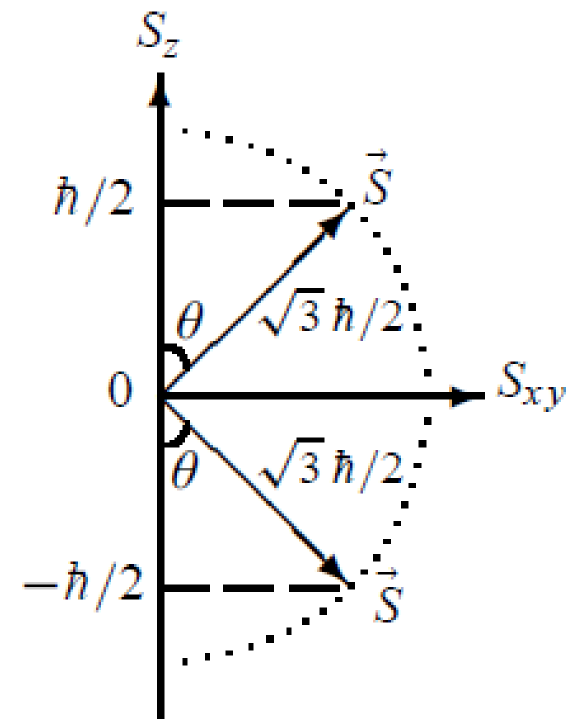
$$Y_{lm}(\pi - \vartheta, \pi + \varphi) \to (-)^l Y_{lm}(\vartheta, \varphi)$$

A.12. Spin

Some particles, like electron or nucleons have an intrinsic angular momentum called spin, which takes two possible values of the projection, i.e. according the general relation 2s+1=2, the spin has a half integer value, s=1/2.

The spin was evidenced by the Stern and Gerlach experiment (1922), where a beam of silver atoms passes through an inhomogeneous magnetic field and it splits into two components, corresponding to the two projections of the spin.





The 2 x 2 spin operator is given by:

$$\hat{\mathbf{S}} = \frac{1}{2}\hbar\hat{\boldsymbol{\sigma}}$$

in terms of the Pauli matrices:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The spin wave function has two components and it is called spinor. The following relations are satisfied:

$$\hat{\mathbf{S}}^2 \chi_{\frac{1}{2}m} = s(s+1)\chi_{\frac{1}{2}m} = \frac{3}{4}\chi_{\frac{1}{2}m}$$

$$\hat{S}_z \chi_{\frac{1}{2}m} = \hbar m \chi_{\frac{1}{2}m}\ ; \quad m = \pm\frac{1}{2}$$

A.13. Isospin

The isospin operator is proportional to the Pauli matrices acting in the "isospin space":

$$\hat{\mathbf{T}} = \frac{1}{2}\hat{\boldsymbol{\sigma}}$$

It describes the proton and neutron as m=±1/2 projections of the nucleonic state

$$\pi(proton) \rightarrow T_z = +\frac{1}{2}$$

$$\nu(neutron) \rightarrow T_z = -\frac{1}{2}$$

In light nuclei with N=Z the total isospin T is conserved and in heavy nuclei with N>Z the third component is conserved $T_z=(N-Z)/2$

A.14. Addition of angular momenta

Angular momenta $\mathbf{j}_1$ and $\mathbf{j}_2$ can be added $\hat{\mathbf{j}} = \hat{\mathbf{j}}_1 + \hat{\mathbf{j}}_2$

where each momentum squared satisfies:

$$\hat{\mathbf{j}}^2 | jm\rangle = j(j+1) | jm\rangle$$

$$\hat{\mathbf{j}}_k^2 | j_k m_k\rangle = j_k(j_k+1) | j_k m_k\rangle; \_\_ k = 1,2$$

The total wave function is given by
the following superposition of products
between the two wave functions:

$$| jm\rangle = \sum_{m_1+m_2=m} \langle j_1 m_1; j_2 m_2 | jm\rangle | j_1 m_1\rangle | j_2 m_2\rangle$$

$$\equiv \left(| j_1\rangle \otimes | j_2\rangle \right)_{jm}$$

where the transformation matrix elements are called
Clebsch-Gordan (CG) coefficients.
The possible values of the total spin obey the triangle rule:

$$| j_1 - j_2 | \le j \le j_1 + j_2$$

A.15. Spin-orbit coupling

is a particular case of the angular momentum addition.
In nuclei the nucleon moves in a mean field
created by the other nucleons.
Its angular momentum is added with the intrinsic spin to the total spin:

$$\hat{\mathbf{j}} = \hat{\mathbf{l}} + \hat{\mathbf{s}}$$

The total wave function is called spin-orbit harmonics and
it can be written by using the ket notation, or the coordinate form:

$$|\, jm\rangle = \sum_{m_l + m_s = m} \left\langle l m_l ; \frac{1}{2} m_s \mid jm \right\rangle |\, l m_l\rangle \left| \frac{1}{2} m_s \right\rangle \Leftrightarrow$$

$$\psi_{jm}(\mathbf{r}, \mathbf{s}) = R_l(r) \sum_{m_l + m_2 = m} \left\langle l m_l ; \frac{1}{2} m_s \mid jm \right\rangle Y_{lm_l}(\vartheta, \varphi) \chi_{\frac{1}{2} m_s}(\mathbf{s})$$

where $R_l(r)$ is the radial wave function and Y_l angular harmonics.
Total spin has two values:

$$j = l \pm \frac{1}{2}$$

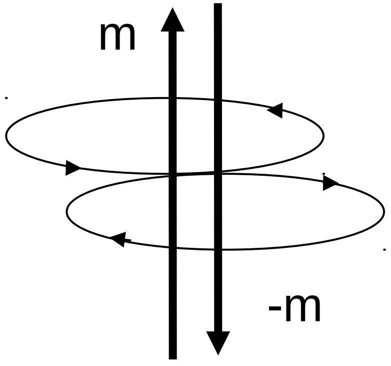
A. 16. Pairing coupling

In nuclear physics one of the fundamental interaction modes
is the pairing coupling between two nucleons.
The wave function is given by the
coupling to the total angular momentum 0:

$$\Psi(\mathbf{r}_1,\mathbf{r}_2) = \left[\psi_j(\mathbf{r}_1)\otimes\psi_j(\mathbf{r}_2)\right]_0 = \frac{1}{\sqrt{2j+1}}\sum_{m=-j}^{j}(-)^{j-m}\psi_{jm}(\mathbf{r}_1)\psi_{j-m}(\mathbf{r}_2)$$

where $\psi_{jm}(\mathbf{r}_\mathbf{k})$ is the wave function of the k-th nucleon in the spin-orbit coupling.

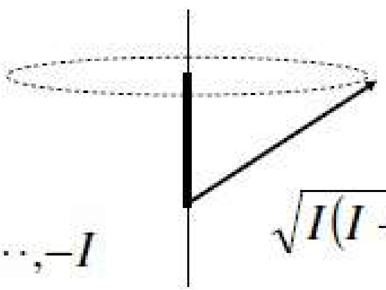
The pictorial representation
of the pairing coupling is given
by two nucleons rotating
in opposite directions



A.17. Spin of nuclei

Nuclear Angular Momentum

- For many applications, the nucleus behaves as a single entity with an angular momentum I
- I is referred to as the nuclear 'SPIN'
- Can include both orbital and spin angular momentum



$m_I \hbar$

$m_I = +I, +(I-1), \cdots, -I$

$\sqrt{I(I+1)}\hbar$

$2I+1$ values

- p and n are fermions $s = \frac{1}{2}$
- Orbital angular momentum is an integer $l = 0, 1, 2 \cdots$
- The total angular momentum of a nucleus is the vector sum of the intrinsic spin and orbital angular momentum of its nucleons

j-j coupling (A > 10)

$$\vec{I} = \sum_i \left(\vec{l}_i + \vec{s}_i \right) = \sum_i \vec{j}_i$$

L-S coupling (A < 10)

$$\vec{I} = \vec{L} + \vec{S}$$

$$\sum_i \vec{l}_i = \vec{L} \quad \& \quad \sum_i \vec{s}_i = \vec{S}$$

- Nuclei with an even number of nucleons (even-A) have

  I = integer

- Nuclei with an odd number of nucleons (odd-A) have

  I = half-integer

- Even-Z, even-N nuclei have $I = 0$

- A consequence of nucleon pairing

- In odd-A nuclei, the nuclear spin is (predominantly) that of the odd nucleon (p or n)

- Odd-Z, odd-N nuclei have

$$\vec{I} = \vec{j}_p + \vec{j}_n$$

- Coupling of the angular momenta of the 'extra' p and 'extra' n.

- Ground state is usually that with

  $s_p \& s_n$ parallel

A.18. Electric transitions

Electric transitions in nuclei are described by
the following multipole transition operator:

$$\hat{\mathbf{Q}}_{\lambda\mu} = r^{\lambda} Y_{\lambda\mu}(\vartheta, \varphi)$$

Quadrupole operator corresponds to the multipolarity *λ=2*
and octupole operator corresponds to the multipolarity *λ=3.*

The probability of electric transitions in nuclei is given by
the reduced transition probability *B(Eλ),* which is proportional
to the modulus squared of the matrix element
between initial (right) and final states (left),
averaged on initial states and summed over final states:

$$B(E\lambda : J_i \to J_f) \propto \frac{1}{2J_i + 1} \sum_{M_i} \sum_{M_f \mu} \left| \left\langle \Psi_{J_f M_f} \,|\, \hat{\mathbf{Q}}_{\lambda\mu} \,|\, \Psi_{J_i M_i} \right\rangle \right|^2$$

Electric transitions are measured in **Weiskopf units (W.u.)**
defined by the transition of one proton with a constant
wave function inside the nuclear region of the radius R

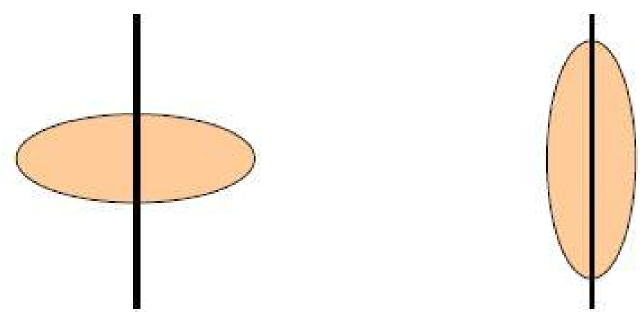
Electric quadrupole momentum

Electric quadrupole momentum of the proton is proportional to expectation value of the the quadrupole operator with the projection *m=0* in the intrinsic system of coordinates, connected to the ellipsoid describing a deformed nucleus

$$\hat{\mathbf{Q}}_{20} = r^2 Y_{20}(\vartheta, \varphi)$$
$$\propto \left(3z^2 - r^2\right)$$

$$eQ = e\int \psi^* \left(3z^2 - r^2\right)\psi \, d\tau$$

- Spherical $Q = 0$
- Planar $Q = -\left\langle r^2 \right\rangle$
- Units of Q = m^2
- 1 barn (b) = 10^{-28} m^2

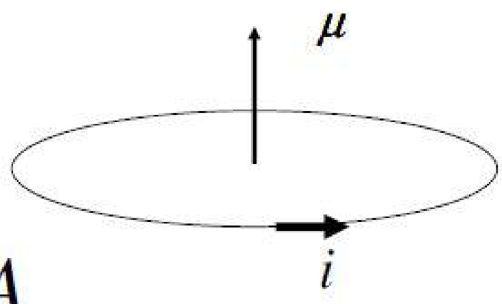


$Q < 0$ Oblate

$Q > 0$ Prolate

A.19. Magnetic dipole moment

Classical magnetic dipole momentum is the product between the intensity of a circular current and the surrounded area and it is proportional to the angular momentum.



$$\mu = iA$$

In quantum mechanics this relation holds between the corresponding operators and the coefficient between them is called Bohr magneton

Relate to angular momentum $$\vec{\mu} = \frac{e\hbar}{2m}\vec{l}$$

$$\hat{\boldsymbol{\mu}} = \mu_B \hat{\mathbf{l}}$$

Atomic moment >> Nuclear moment

$m_e \rightarrow \mu_B$ "Bohr Magneton"

$m_p \rightarrow \mu_N$ "Nuclear Magneton"

$$\frac{m_p}{m_e} = 1837$$

Nuclear g-factors

Angular momentum → Magnetic moment

$$\vec{\mu}_l = g_l \mu_N \vec{l}$$

Gyromagnetic ratio

$g_l = 1$ proton

$g_l = 0$ neutron

$$\vec{\mu}_s = g_s \mu_N \vec{s}$$

$s = \frac{1}{2}$ for protons and neutrons

Expect $g_s = 2$ for elementary (Dirac) charged particles

$$g_s \approx \begin{cases} 5.5857 & \text{proton} \\ -3.8261 & \text{neutron} \end{cases}$$

Summary
Basic physical observables in nuclear structure:

energy level: E

angular momentum parity : $J^{\pm}$

electric decay rate: $B(E\lambda{:}J_i\text{->}J_f)$

electric quadrupole momentum: eQ_{20}

magnetic dipole momentum: μ

A.20. Schrodinger equation

The time evolution of the wave function
is described by the Schrodinger equation:

$$\hat{\mathbf{H}}\psi = i\hbar \frac{\partial \psi}{\partial t}$$

where the Hamiltonian is given by the sum between the kinetic and potential operators.

For a stationary wave function

$$\psi(t, \mathbf{r}) = e^{-i\frac{E}{\hbar}t}\phi(\mathbf{r})$$

one obtains the following stationary equation:

$$\hat{\mathbf{H}}\phi = E\phi$$

The eigenstates correspond to observable energies

$$\left\langle \phi_n \,|\, \hat{\mathbf{H}} \,|\, \phi_n \right\rangle = E_n \phi_n$$

A.21. One dimensional harmonic oscillator

The Hamiltonian is given by

$$\hat{\mathbf{H}} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 = \frac{\hbar\omega}{2}\left(-\frac{d^2}{dq^2} + q^2\right) \equiv \frac{\hbar\omega}{2}\left(\hat{\mathbf{p}}^2 + \hat{\mathbf{q}}^2\right)$$

where we defined the reduced coordinate: $q = \dfrac{x}{x_0}$

in terms of the ho length parameter: $x_0 \equiv \sqrt{\dfrac{\hbar}{m\omega}}$

We also defined the reduced conjugate operators:

$$\hat{\mathbf{q}} = q$$

$$\hat{\mathbf{p}} = -i\frac{d}{dq}$$

satisfying the commutation rule: $\left[\hat{\mathbf{q}}, \hat{\mathbf{p}}\right] = i$

By introducing the creation and annihilation boson operators:

$$\hat{\mathbf{b}}^{+} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{q}} - i\hat{\mathbf{p}}\right)$$

$$\hat{\mathbf{b}} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{q}} + i\hat{\mathbf{p}}\right)$$

with the commutator:

$$\left[\hat{\mathbf{b}}, \hat{\mathbf{b}}^{+}\right] = 1$$

one obtains:

$$\hat{\mathbf{H}} = \hbar\omega\left(\hat{\mathbf{N}} + \frac{1}{2}\right)$$

where the occupation number operator is:

$$\hat{\mathbf{N}} \equiv \hat{\mathbf{b}}^{+}\hat{\mathbf{b}}$$

Due to the fact that the two operators commute:

$$\left[\hat{\mathbf{H}}, \hat{\mathbf{N}}\right] = 0$$

they have a common set of eigenstates:

$$\hat{\mathbf{N}}\left|n\right\rangle = n\left|n\right\rangle$$

$$\hat{\mathbf{H}}\left|n\right\rangle = E_n\left|n\right\rangle = \hbar\omega\left(n + \frac{1}{2}\right)\left|n\right\rangle$$

The equation for the ground state with n=0, i.e.with the energy $E_0=\hbar\omega/2$

$$\hat{\mathbf{H}}\psi_0(q)=\frac{\hbar\omega}{2}\left(-\frac{d^2}{dq^2}+q^2\right)\psi_0(q)=\frac{\hbar\omega}{2}\psi_0(q)$$

is satisfied by the following state normalized in the x coordinate:

$$\psi_0(q)=\frac{1}{\sqrt{\sqrt{\pi}\,x_0}}e^{-\frac{q^2}{2}}$$

The following commutators can be readily obtained **(home work)**:

(1) $\left[\hat{\mathbf{N}},\hat{\mathbf{b}}^{+}\right]=\hat{\mathbf{b}}^{+}$ $\left[\hat{\mathbf{N}},\hat{\mathbf{b}}\right]=-\hat{\mathbf{b}}$

(2) $\left[\hat{\mathbf{H}},\hat{\mathbf{b}}^{+}\right]=\hbar\omega\hat{\mathbf{b}}^{+}$ $\left[\hat{\mathbf{H}},\hat{\mathbf{b}}\right]=-\hbar\omega\hat{\mathbf{b}}$

(1) implies that **b⁺** creates and **b** annihilates a phonon:

$$\hat{\mathbf{N}}\hat{\mathbf{b}}^{+}|n\rangle=(n+1)\hat{\mathbf{b}}^{+}|n\rangle$$

$$\hat{\mathbf{N}}\hat{\mathbf{b}}|n\rangle=(n-1)\hat{\mathbf{b}}|n\rangle$$

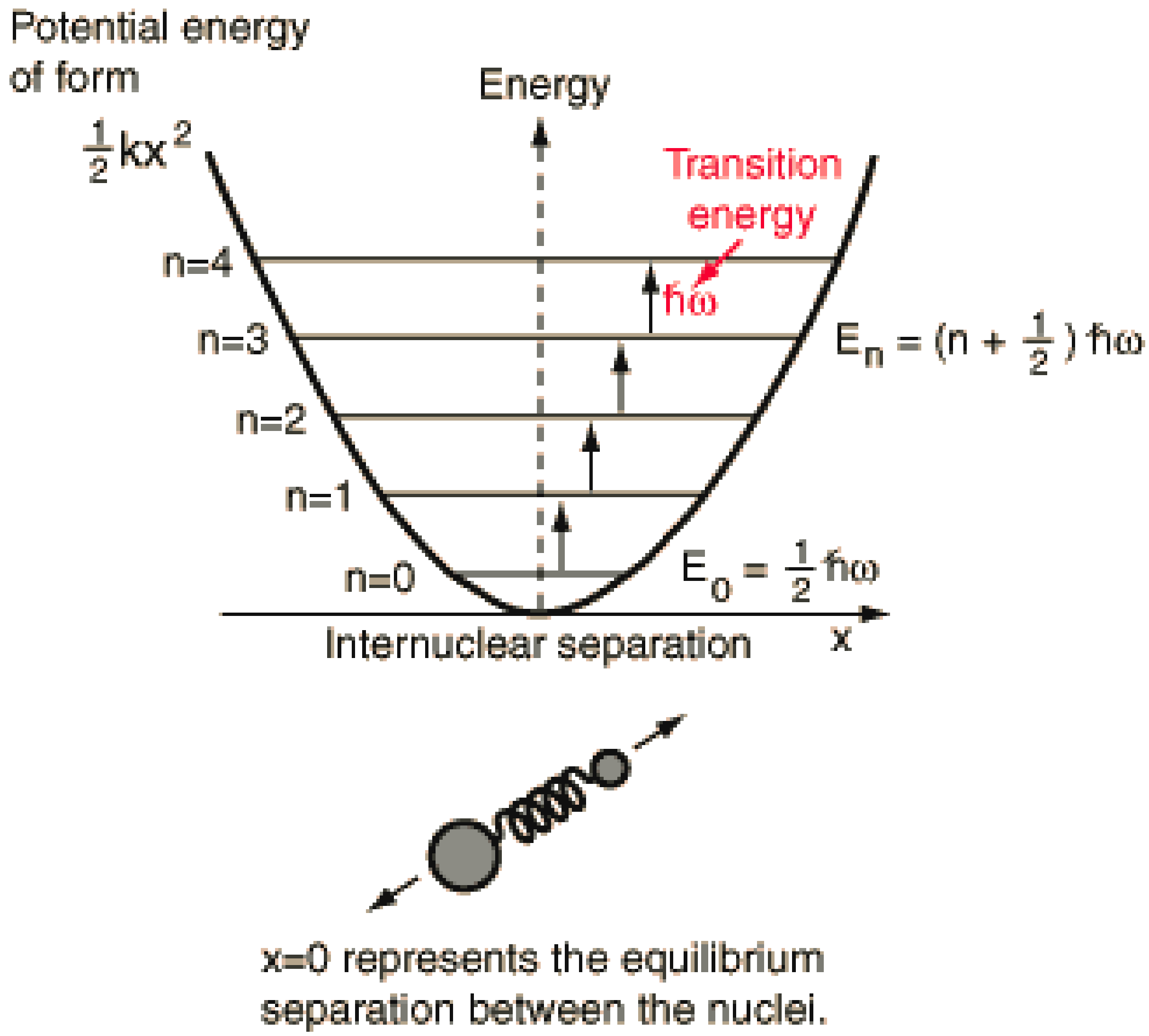
Thus, the number n defining the ho spectrum, is an integer number, defining the number of oscillator quanta (or phonons)

(2) implies that **b⁺** creates and **b** annihilates the energy *ħω* of the phonons:

$$\hat{\mathbf{H}}\hat{\mathbf{b}}^{+}|n\rangle=(E_n+\hbar\omega)\hat{\mathbf{b}}^{+}|n\rangle$$

$$\hat{\mathbf{H}}\hat{\mathbf{b}}|n\rangle=(E_n-\hbar\omega)\hat{\mathbf{b}}|n\rangle$$

Spectrum of the ho oscillator



ho wave functions

are given by:

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}\,2^n n!\,x_0}} e^{-\frac{x^2}{2x_0^2}} H_n\left(\frac{x}{x_0}\right)$$

in terms of Hermite polinomials, which are defined as follows:

$$H_n(y) = (-)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$$

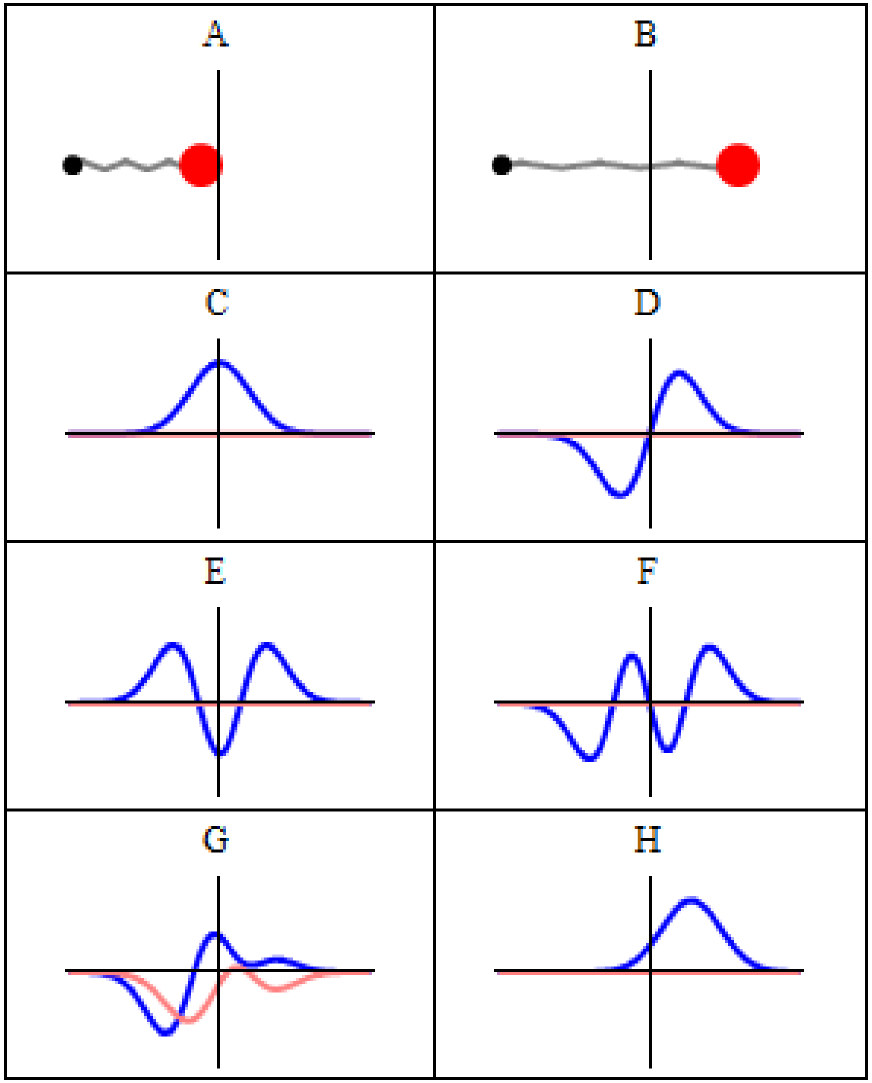
The following relations hold **(home work)**:

$$\hat{\mathbf{b}}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{\mathbf{b}}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$|n\rangle = \frac{\left(\hat{\mathbf{b}}^+\right)^n}{\sqrt{n!}}|0\rangle$$

Shape of the ho wave functions



A.22. Dinuclear systems

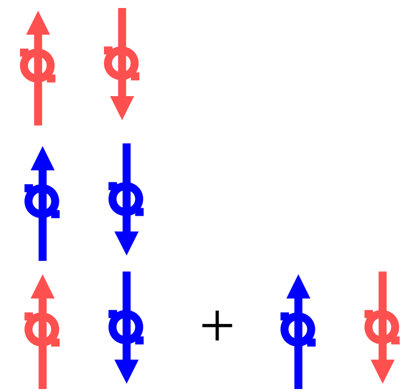
Fundamental degrees of freedom in nuclei are protons and neutrons. Proton-proton ($\pi\pi$), neutron-neutron ($\nu\nu$) & proton-neutron ($\pi\nu$) *systems* have the following symmetries:

1) Isovector systems: T=1, S=0

T_z=+1 $\qquad \pi\pi$

T_z=-1 $\qquad \nu\nu$

T_z= 0 $\qquad \frac{1}{\sqrt{2}}\left(\pi\nu + \nu\pi\right)$

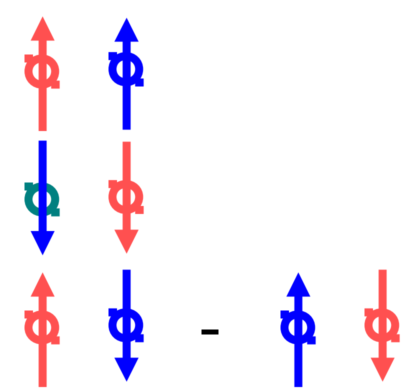


2) Isoscalar systems: T=0, S=1

S_z=+1 $\qquad \pi\nu$

S_z=-1 $\qquad \nu\pi$

S_z= 0 $\qquad \frac{1}{\sqrt{2}}\left(\pi\nu - \nu\pi\right)$



A.23. Deuteron

is the simplest bound nuclear system with T=0, S=1

1 proton + 1 neutron

Charge = +1, Mass ~ 2 u

$m_p = 1.007276$ u

$m_n = 1.008665$ u

$m_p + m_n = 2.015941$ u

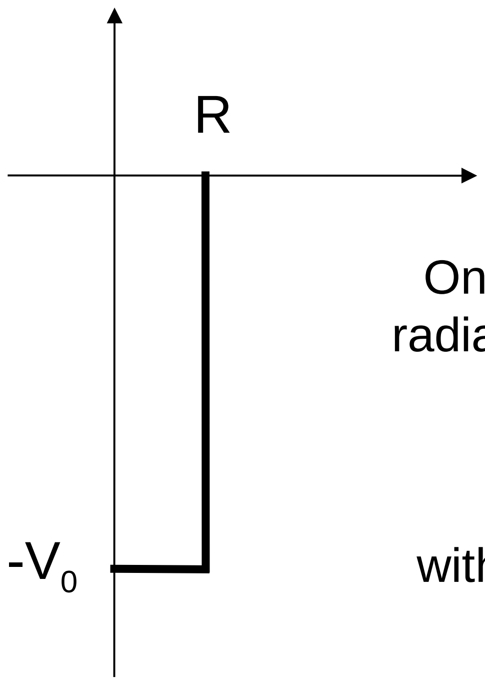
$m_d = 2.013553$ u

$$m_d < m_p + m_n$$

The system is BOUND

Spherical square well

is a good approximation of the deuteron



$$V(r) = -V_0 = -35MeV, \_\_(r \le R = 2\,fm)$$
$$= 0 \_\_(r > R)$$

One obtains the radial Schrodinger equation:

$$\frac{d^2 f_l(r)}{dr^2} + \left[-\frac{l(l+1)}{r^2} + k_0^2 \right] f_l(r) = 0$$

with momentum:

$$k_0 = \frac{\sqrt{2m(V_0 + E)}}{\hbar} = \frac{\sqrt{2mc^2(V_0 + E)}}{\hbar c}$$

For *l=0* one obtains the following equation:

$$\frac{d^2 f_0(r)}{dr^2} + k_0^2 f_0(r) = 0$$

The solution vanishes at *r=0* and
the logarithmic derivative should be continuous at *r=R*

E=-|*E*|<*0*: bound states

Internal solution $$f_0(r) = A\sin k_0 r, \_\_\_\_ r \le R$$

External solution $$f_0(r) = Be^{-kr}, \_\_\_\_ r > R$$

Bound-state momentum $$k^2 = \frac{2m|E|}{\hbar^2}$$

The energy spectrum is discrete and
it is given by the equation expressing
the continuity of logarithmic derivatives:

$$\frac{f_0^{'}(R)}{f_0(R)} = k_0 tg k_0 R = -k$$

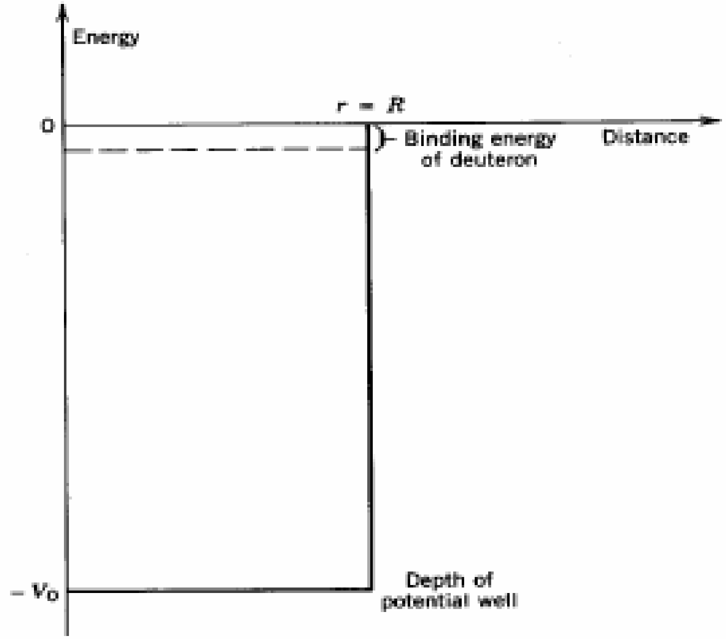
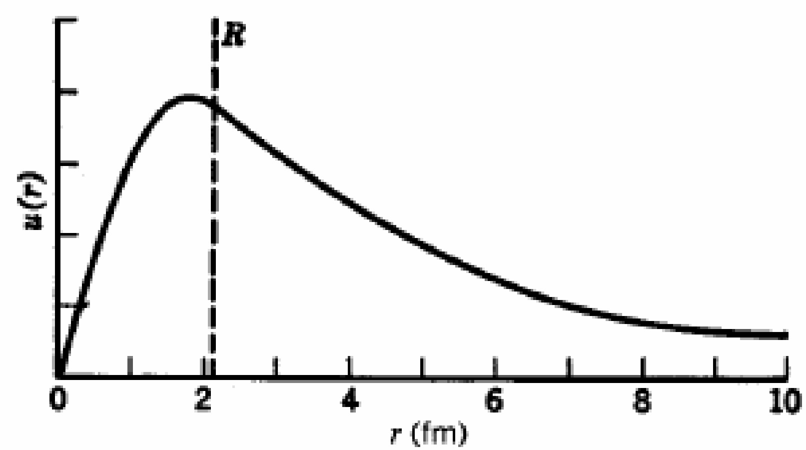


Figure 4.1 The spherical square-well potential is an approximation to the nuclear potential. The depth is $-V_0$, where V_0 is deduced to be about 35 MeV. The bound state of the deuteron, at an energy of about -2 MeV, is very close to the top of the well.



Spin & magnetic moment of the deuteron

- d = p + n

$$S_p = \frac{1}{2} \qquad S_n = \frac{1}{2}$$

$$\vec{S}_d = \vec{S}_p + \vec{S}_n$$

- possibilities are

$$S_d = 0 \quad (\uparrow\downarrow) \qquad S_d = 1 \quad (\uparrow\uparrow)$$

- Experimentally, the deuteron has only 1 bound state with $S_d = 1$
- Therefore, the strong interaction is spin-dependent !

- Magnetic moment

$$l = 0 \quad \rightarrow \quad \mu_{orb} = 0$$

$$\vec{\mu}_d = \vec{\mu}_p + \vec{\mu}_n = 0.8798\mu_N$$

$$\mu_{\text{expt}} = 0.8574\mu_N$$

$$\psi = a_s\psi_s + a_d\psi_d$$

$l = 0$ state (→ $a_s\psi_s$) $l = 2$ state (→ $a_d\psi_d$)

$$a_s^2 \approx 0.96$$

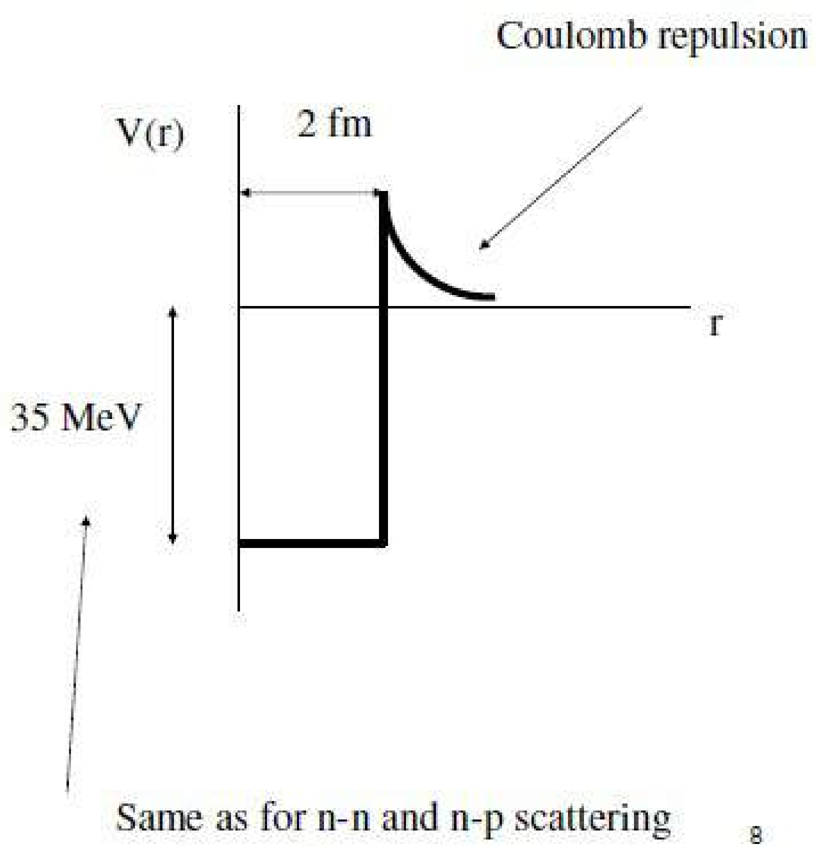
$$a_d^2 \approx 0.04$$

A.24. Nuclear two body potential

A schematic description is given by a spherical well surrounded by Coulomb repulsion for the proton-proton system

Nuclear forces

- Proton-proton scattering



Strong interaction between nucleons is mediated by the exchange of virtual mesons

- 1947 – Powell et al. discovery of Pions (pi-mesons)

$$m(\pi^+) = m(\pi^-) = 140\ MeV/c^2 \quad (274 m_e)$$

$$\tau_\pi \sim 26 ns$$

- 1950 – Moyer et al.

$$m(\pi^o) = 135\ MeV/c^2 \quad (264 m_e)$$

- Pions have $s = 0$ so pion exchange between nucleons will conserve angular momentum

- Pion exchange mechanisms

$$n \rightarrow n + \pi^o \quad \Leftrightarrow \quad \pi^o + p \rightarrow p$$

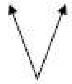
$$n \rightarrow p + \pi^- \quad \Leftrightarrow \quad \pi^- + p \rightarrow n$$

$$p \rightarrow n + \pi^+ \quad \Leftrightarrow \quad \pi^+ + n \rightarrow p$$

- Nucleons swap identities in about 50% of the events

- Pion exchange could also provide an 'explanation' for the magnetic moment of the uncharged neutron

$$n \rightarrow p + \pi^-$$



Yukawa potential

describes the interaction between nucleons
by the exchange of virtual mesons

$$V(r) = g \frac{e^{-r/R}}{r/R}$$

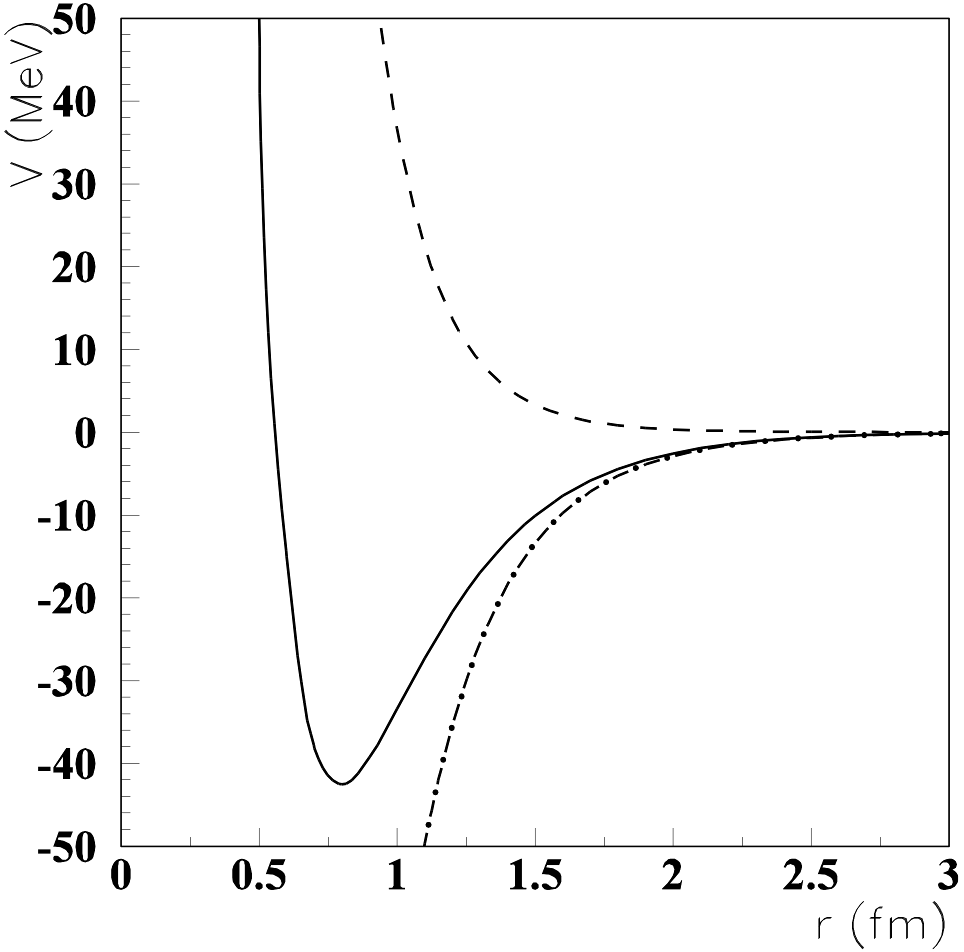
where *g* is the coupling constant
and the range of the interaction
is given by the pion mass:

$$R = \frac{\hbar c}{m} = 1.4\, fm$$

Notice that in the case of the electromagnetic interaction the range of the interaction is infinite because the photon mass is zero.

Effective radial nuclear interaction

extracted from scattering between heavy ions
can be fitted by a superposition of two Yukawa potentials



$$V_C(\mathbf{r}) = g_1 \frac{e^{-r/R_1}}{r/R_1} + g_2 \frac{e^{-r/R_2}}{r/R_2}$$

$$g_1 = +7999 MeV; R_1 = 0.25\, fm$$

$$(dashed)$$

$$g_2 = -2134 MeV; R_2 = 0.40\, fm$$

$$(dot-dashed)$$

Nuclear interaction potential has three components

$$V(\mathbf{r}) = V_C(\mathbf{r}) + V_S(\mathbf{r}) + V_T(\mathbf{r})$$

1) Central component:
Radial part can be well
described by a superposition
of Yukawa potentials.

$$V_C(\mathbf{r}) = \sum_{k=1,2} g_k \frac{e^{-r/R_k}}{r/R_k}$$

2) Spin-spin component:

$$V_S(\mathbf{r}) = \mathrm{v}_S(\mathbf{r})(\mathbf{s}_1.\mathbf{s}_2)$$

3) Non-central tensorial
component:

$$V_T(\mathbf{r}) = \mathrm{v}_T(\mathbf{r})\left(3\frac{(\mathbf{s}_1.\mathbf{r})(\mathbf{s}_2.\mathbf{r})}{r^2} - (\mathbf{s}_1.\mathbf{s}_2)\right)$$

Summary:
Characteristics of the nuclear force

-very strong (MeV)
-short range between nucleons (fm)
-charge independent
-spin dependent
-interaction contains:
    central component
    spin-spin component
    non-central (tensorial) component
-attractive at large distances
-repulsive at small distances
-the effective radial dependence is given by the sum of
    two Yukawa potentials (one atractive + one repulsive)
    describing the exchange of neutral mesons
    with different masses between nucleons