B. PHENOMENOLOGICAL NUCLEAR MODELS

B.01. Basic concepts of nuclear physics

B.02. Binding energy

B.03. Liquid drop model

B.04. Spherical operators

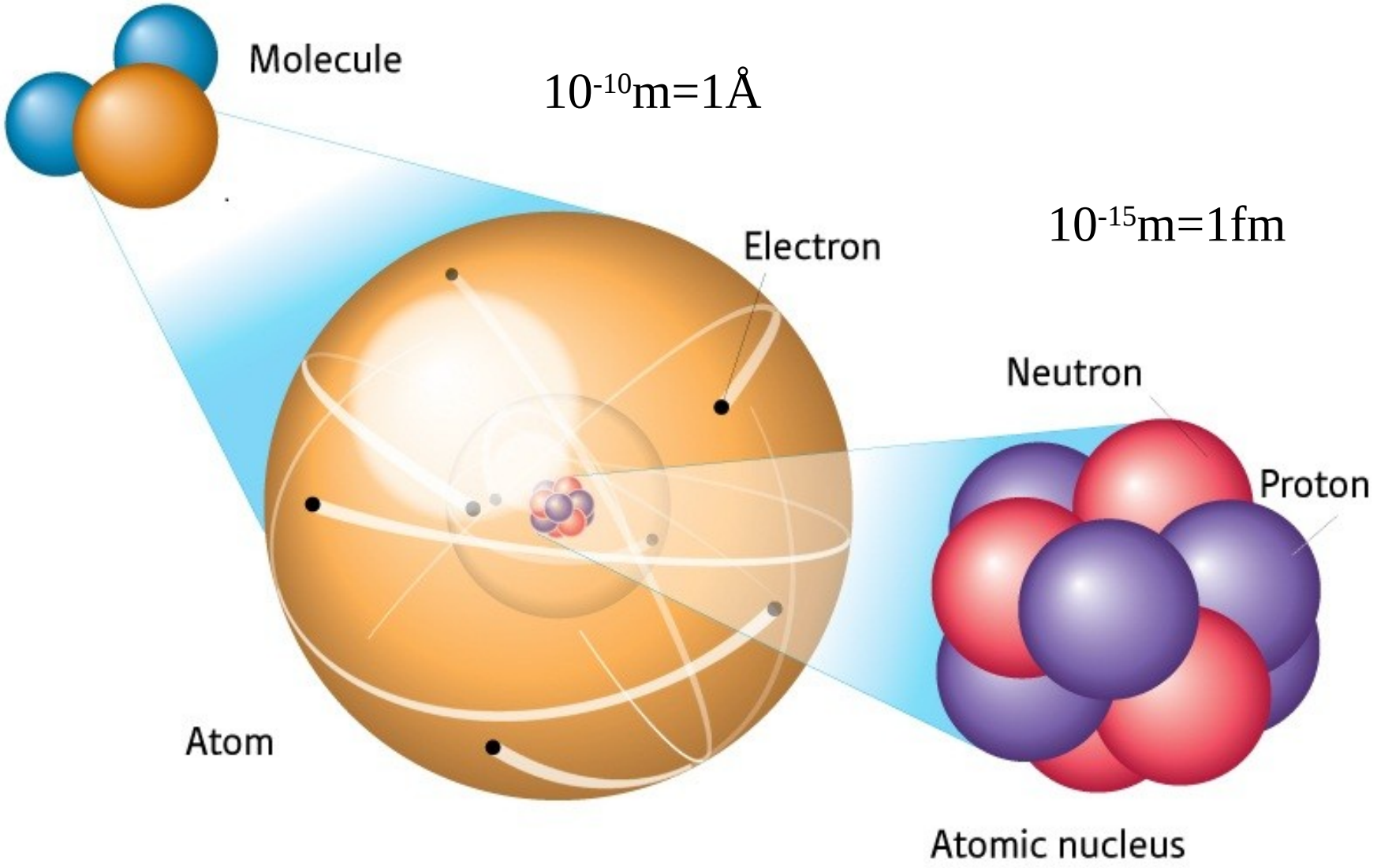
B.05. Bohr-Mottelson model

B.06. Intrinsic system of coordinates

B.07. Interacting Boson Model (optional lecture)

B.08. Giant resonance

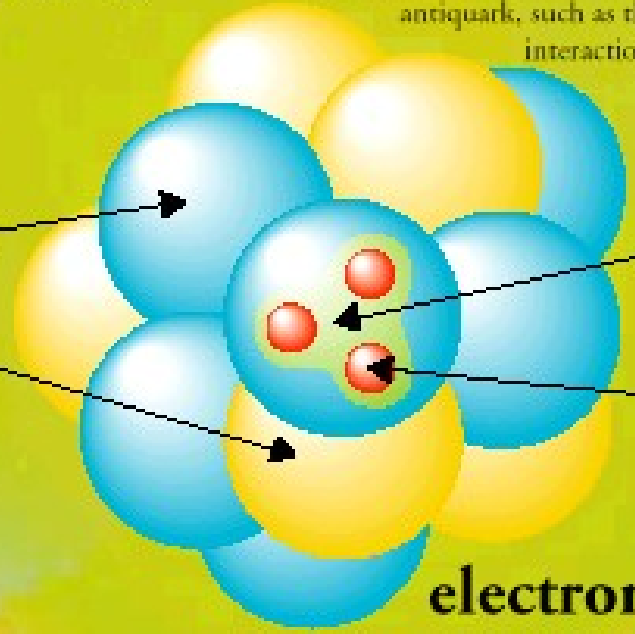
B.01. Basic concepts of nuclear physics



The Nucleus

$(1\text{-}10) \times 10^{-15}$ m

At the center of the atom is a nucleus formed from **nucleons**–protons and neutrons. Each nucleon is made from three **quarks** held together by their strong interactions, which are mediated by gluons. In turn, the nucleus is held together by the **strong** interactions between the gluon and quark constituents of neighboring nucleons. Nuclear physicists often use the exchange of mesons–particles which consist of a quark and an antiquark, such as the **pion**–to describe interactions among the nucleons.



neutron

10^{-15} m

proton

strong field

quark

$<10^{-19}$ m

electromagnetic field

In an atom, **electrons** range around the nucleus at distances typically up to 10,000 times the nuclear diameter. If the electron cloud were shown to scale, this chart would cover a small town.

nucleon: proton or neutron

nuclide: nucleus uniquely specified by

number of protons (Z) and neutrons (N)

mass number: A=Z+N

isotopes: nuclides with the same Z

ex: ^{235}U and ^{238}U

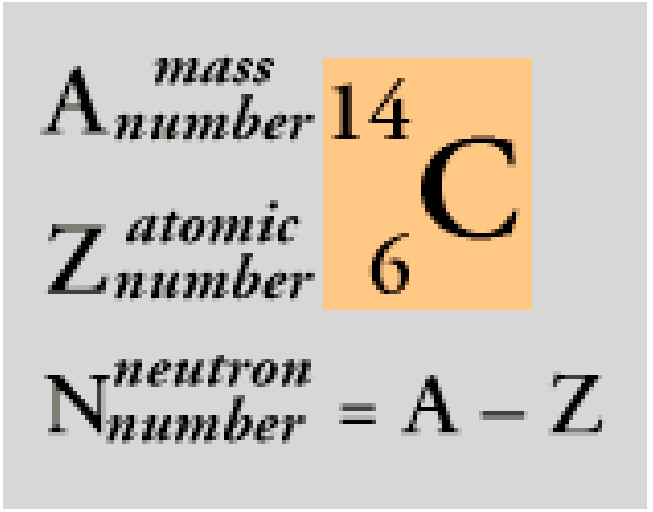
isotones: nuclides with the same N

ex: ^{2}H, ^{3}He

isobars: nuclides with the same A

atomic mass unit: 1u=1/12 m(^{12}C)

=1.66 10^{-27}kg=931.5 MeV/c^2



Units used in nuclear physics

Length

1 fm $=10^{-15}$ m

Energy

1 MeV = 10^{6} eV

1 eV = 1,6 10^{-19} J

Basic constants

Nucleonic mass:

M_N=938,90 MeV/c^2

Planck constant * light velocity:

ħc=197,33 MeV.fm

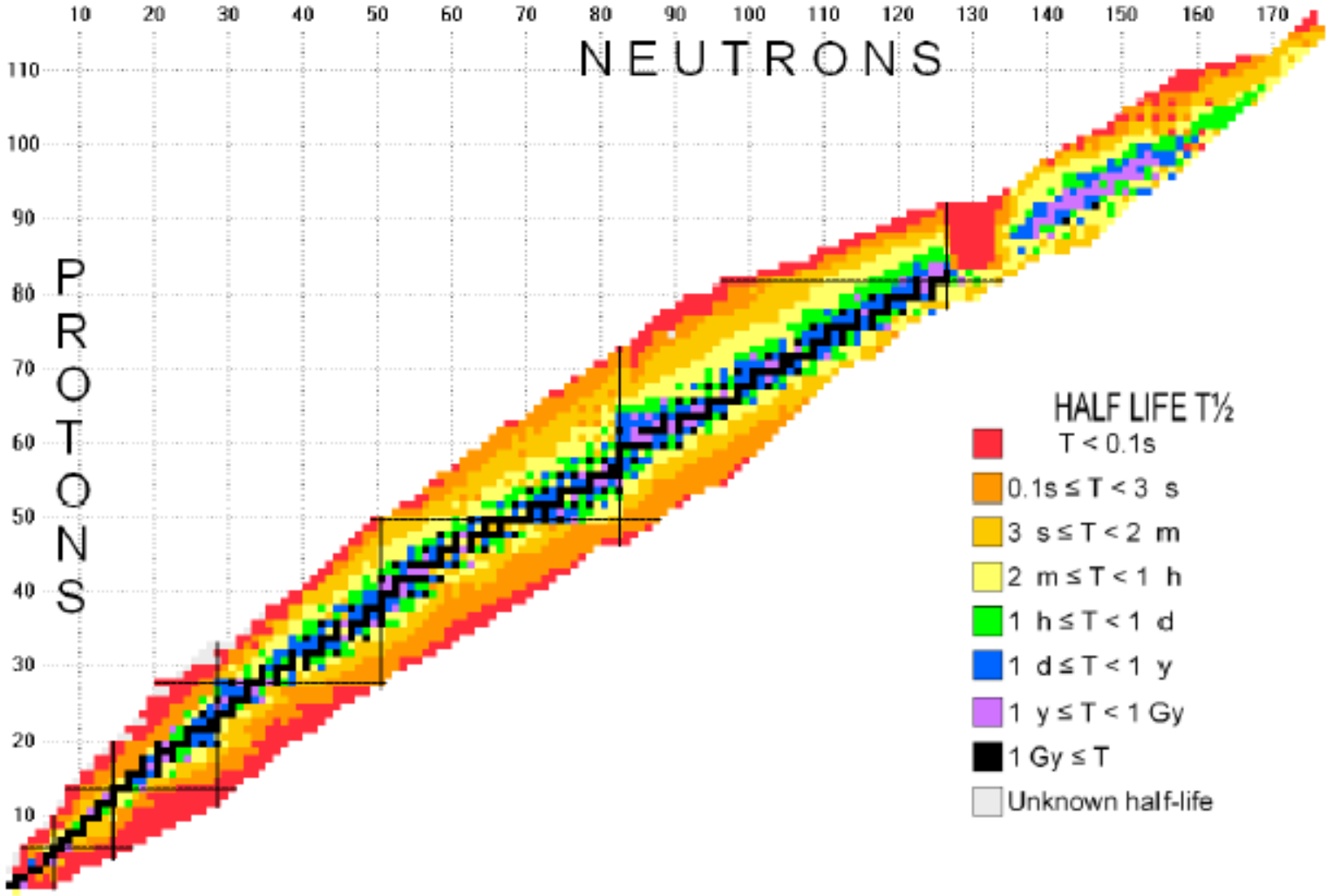
electric charge squared:

e^2=ħc/137=1,44 MeV.fm

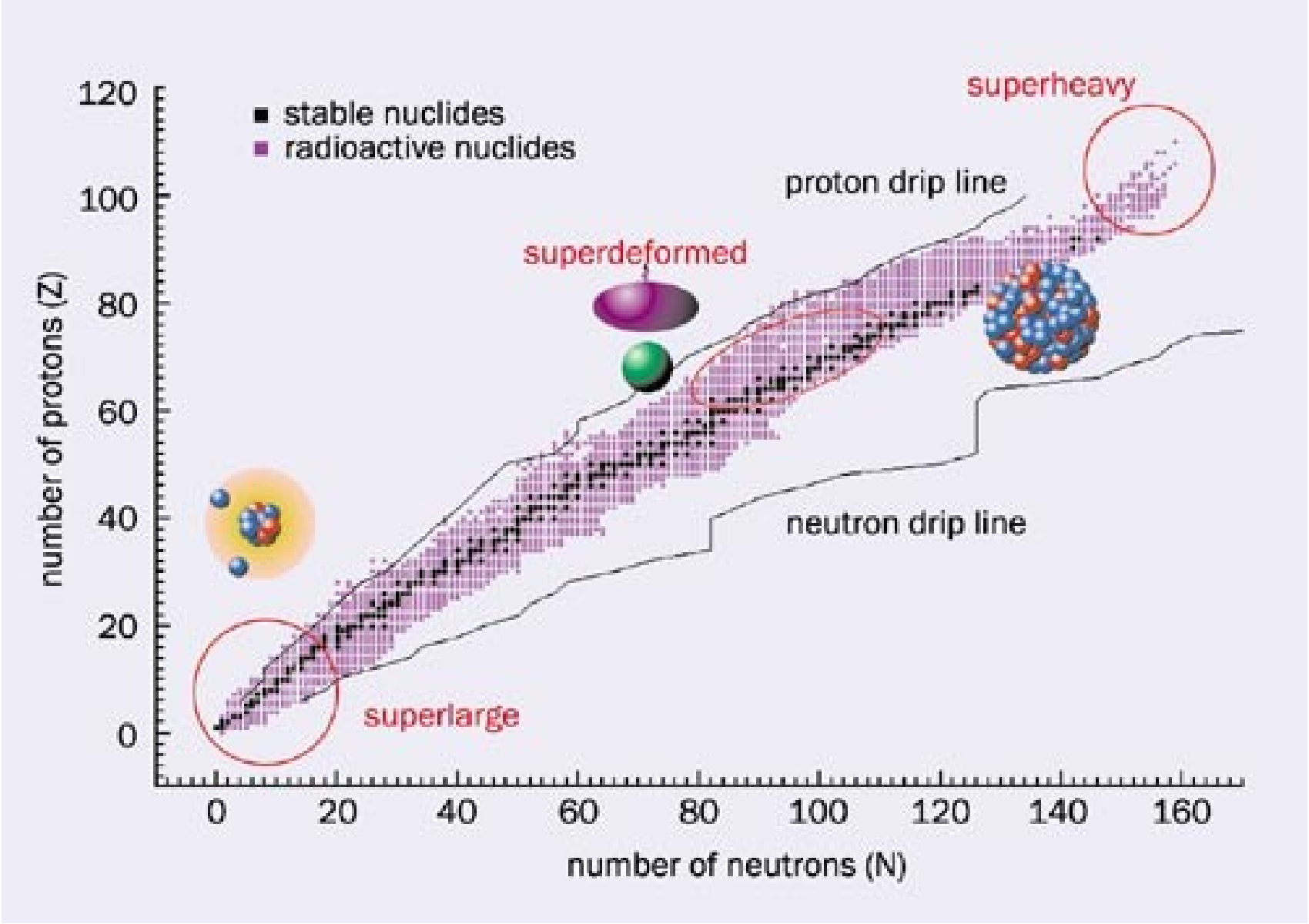
Properties of nucleons

	proton	neutron
mass	1.007276= 938.280 MeV/c^2	1.008665= 939.573 MeV/c^2
charge	+1	0
spin	1/2	1/2
magnetic moment	+2.7928 μ_N	-1.9128 μ_N
parity	+1	+1

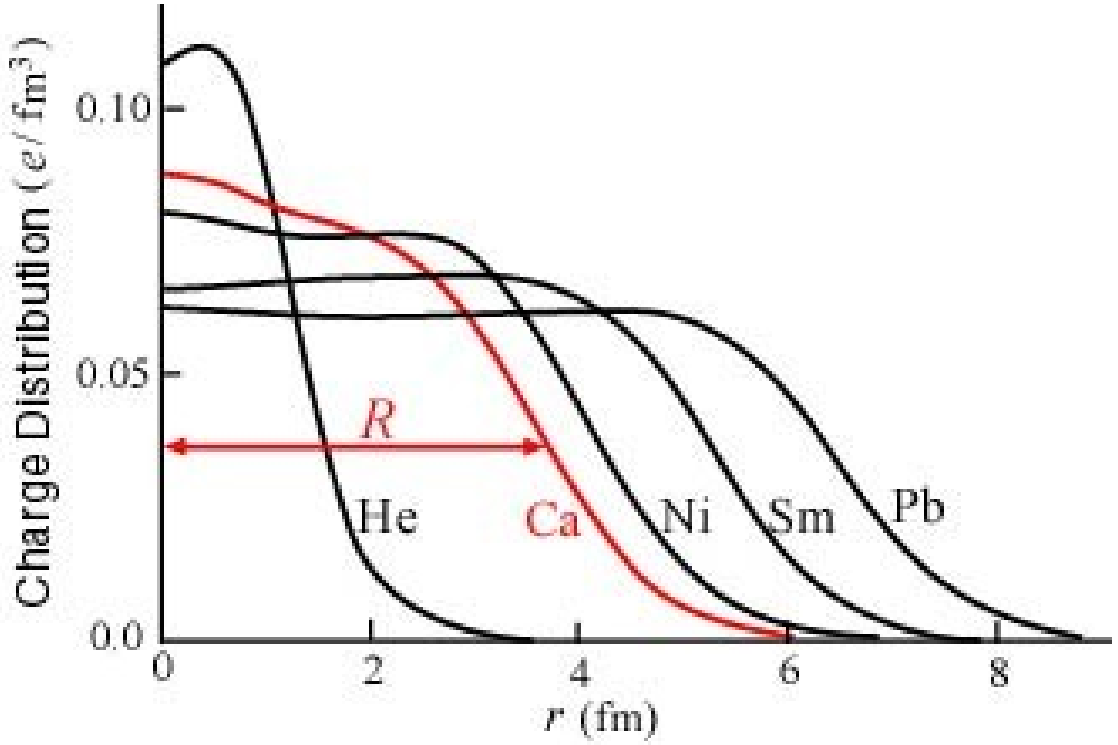
Nuclear chart
stability of nuclei



Limits of stable nuclei
exotic nuclei



Nuclear size
from electron scattering experiments



$$R = r_0 A^{1/3}$$

$$r_0 = 1.2\ fm$$

B.02. Binding energy

$$B = (ZM_p + NM_n + ZM_e\text{-}M)c^2$$

Mass defect

$$\Delta = (M - A)c^2$$

Example

Binding energy

- e.g. ^{16}O (Z = 8 and N = 8)

$M_p = 938.280\ MeV/c^2$
$M_n = 939.573\ MeV/c^2$
$M_e = 0.511003\ MeV/c^2$

$$8M_p + 8M_n + 8M_e = 15026.912\ MeV/c^2$$

$$Atomic\ Mass = 15.994915\ u$$

$$= 15.994915 \times 931.502$$

$$= 14899.295\ \ MeV/c^2$$

$$\Delta M = 15026.912 - 14.899.295$$

$$= 127.617\ MeV/c^2$$

Mass Defect (Δ)

- e.g. ^{16}O (Z = 8 and N = 8)

$$\Delta = \left(m_{nucl.} - A\right)c^2$$

$$\Delta = (M - A)c^2$$

$$= (15.994915 - 16)c^2$$

$$= -0.005085\ c^2\ MeV$$

$$= -0.005085 \times 931.502 = -4.737\ \ MeV$$

Separation Energy

• Neutron Separation Energy = energy to remove one neutron from nucleus

$$= B\left({}^{A}_{Z}X_{N}\right) - B\left({}^{A-1}_{Z}X_{N-1}\right)$$

$$= \left[m_{(A-1)} + m_n - m_A\right]c^2$$

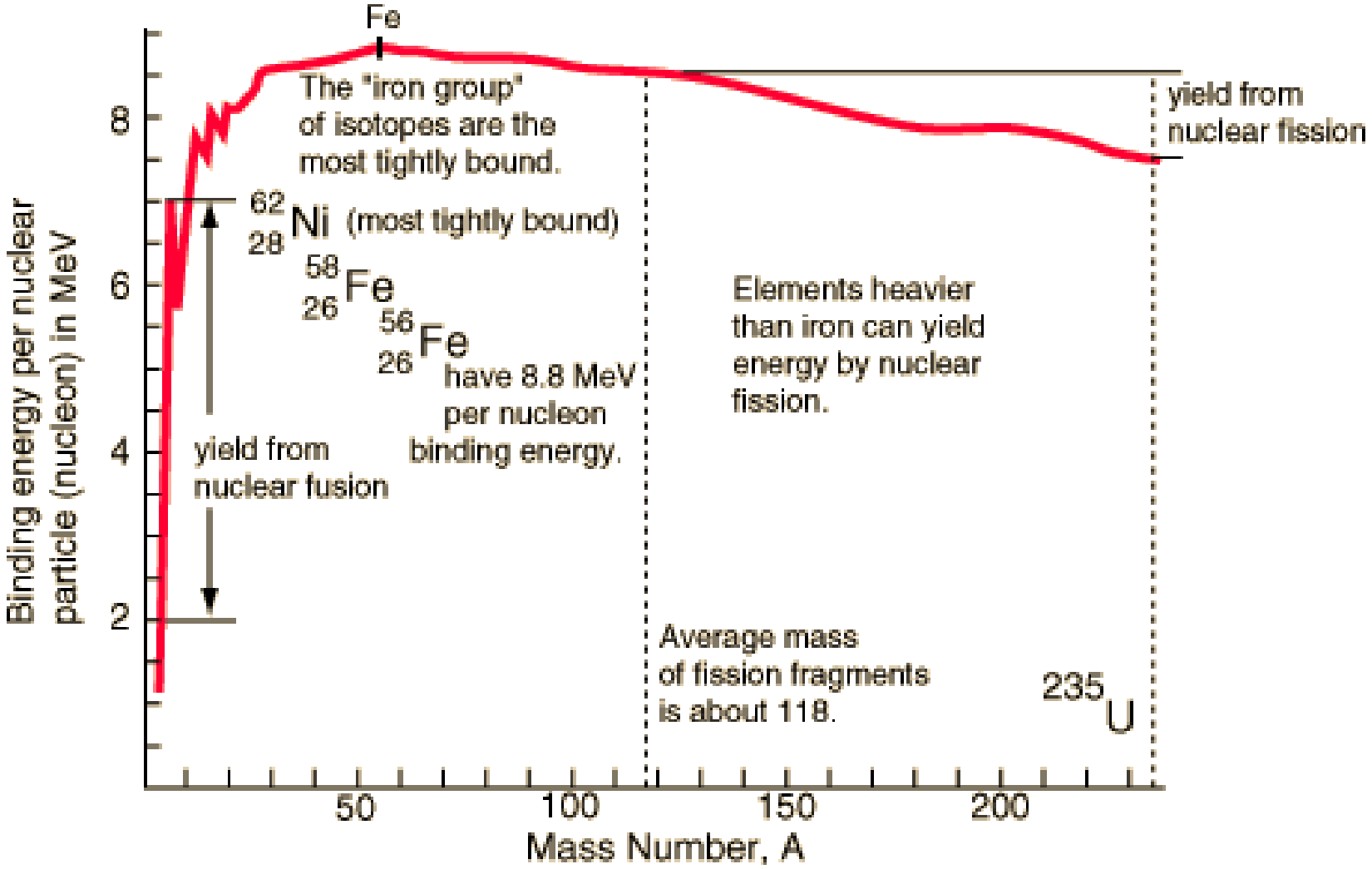
$$e.g.\ {}^{16}_{8}O_8 = \left(\begin{matrix}15.003065 \\ -15.994915\end{matrix}\ +1.00866501\right)c^2$$

$$= 0.016815\ c^2$$

$$= 15.6632\ MeV$$

Binding energy / nucleon: B/A

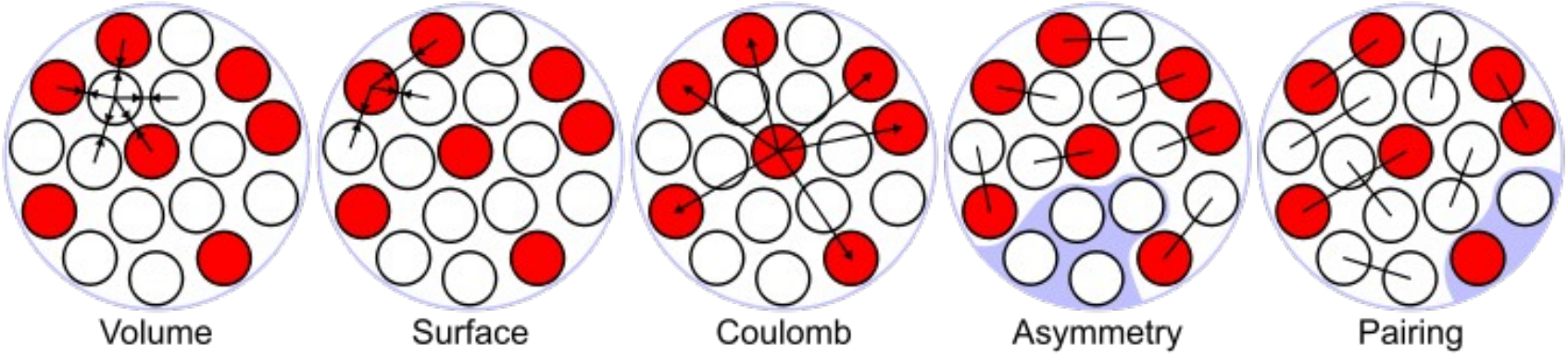
has an almost constant value ≈ 8 MeV / nucleon
expressing the saturation property of nuclear forces
like in a liquid drop



B.03. Liquid drop model

Weizsäcker semiempirical formula (1935)

$$E_B = a_V A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} + \delta(A,Z)$$



Volume term

For $A \geq 30$

$$B/A \approx Const.$$

$$Vol \propto R^3 \therefore \propto A$$

$$B_V = a_V A$$

~ 8 MeV

A nucleon attracts its closest neighbours

Surface term

A nucleon near the surface has fewer neighbours

As A ↑, the surface area ↑ so the number of 'surface nucleons' ↑

Need to reduce Binding Energy

$$Area \propto R^2 \therefore \propto A^{2/3}$$

$$B_S = -a_S A^{2/3}$$

Coulomb repulsion term

- Proton-proton repulsion makes nucleons less tightly bound.
- Long-distance Coulomb repulsion
- Need to reduce Binding Energy
- Z protons : how many p–p pairs ?

$$\frac{Z!}{2!(Z-2)!} = Z(Z-1)/2$$

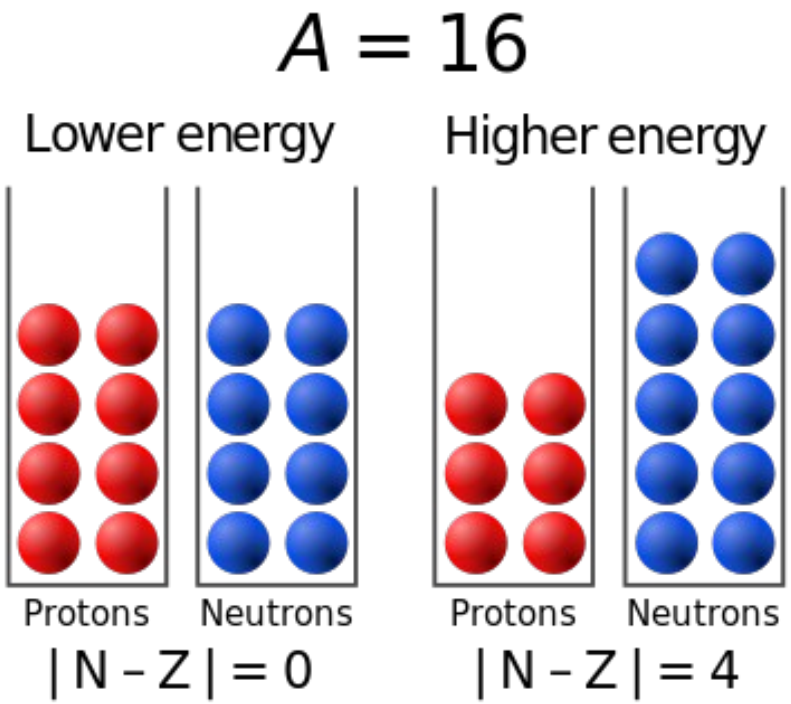
$$Coulomb\ energy \propto \frac{1}{R} \therefore \propto A^{-1/3}$$

$$B_C = -a_C \frac{Z(Z-1)}{A^{1/3}}$$

Symmetry energy

Light, stable nuclei tend to have $Z \sim N \sim A/2$.

As A ↑ there is more p–p Coulomb repulsion so $N > Z$



Imbalance between N and Z

$$B_{Sym} = -a_{Sym}\frac{(A-2Z)^2}{A}$$

Reduces the effect as A ↑

Pairing term

Even–Even nuclei are particularly stable

Z = even & N = even.

Odd–Odd nuclei tend to be unstable

[Later: Shell Model]

$$B_{pair} = \pm\delta$$

Even – Even → +

Odd – Odd → –

Even – Odd → 0

Pairing term

We can see the effect of the pairing term in the Mass Parabolas

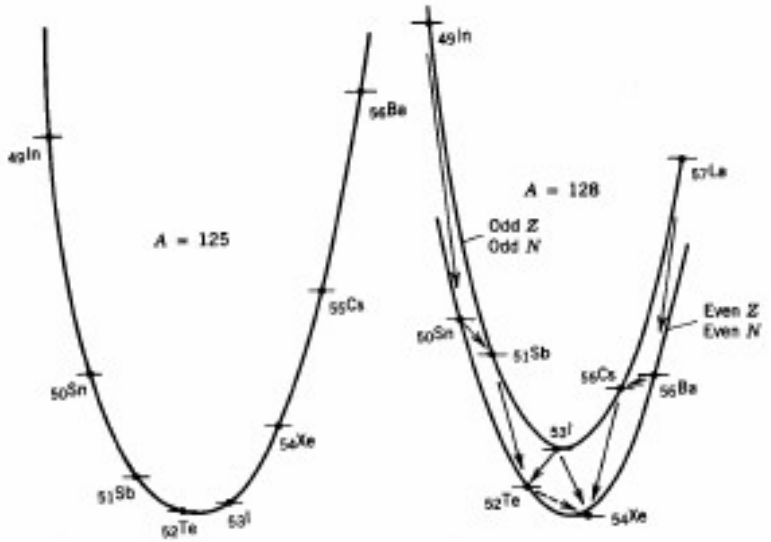


Figure 3.18 Mass chains for $A = 125$ and $A = 128$. For $A = 125$, note how the energy differences between neighboring isotopes increase as we go further from the stable member at the energy minimum. For $A = 128$, note the effect of the pairing term; in particular, ^{128}I can decay in either direction, and it is energetically possible for ^{128}Te to decay directly to ^{128}Xe by the process known as double β decay.

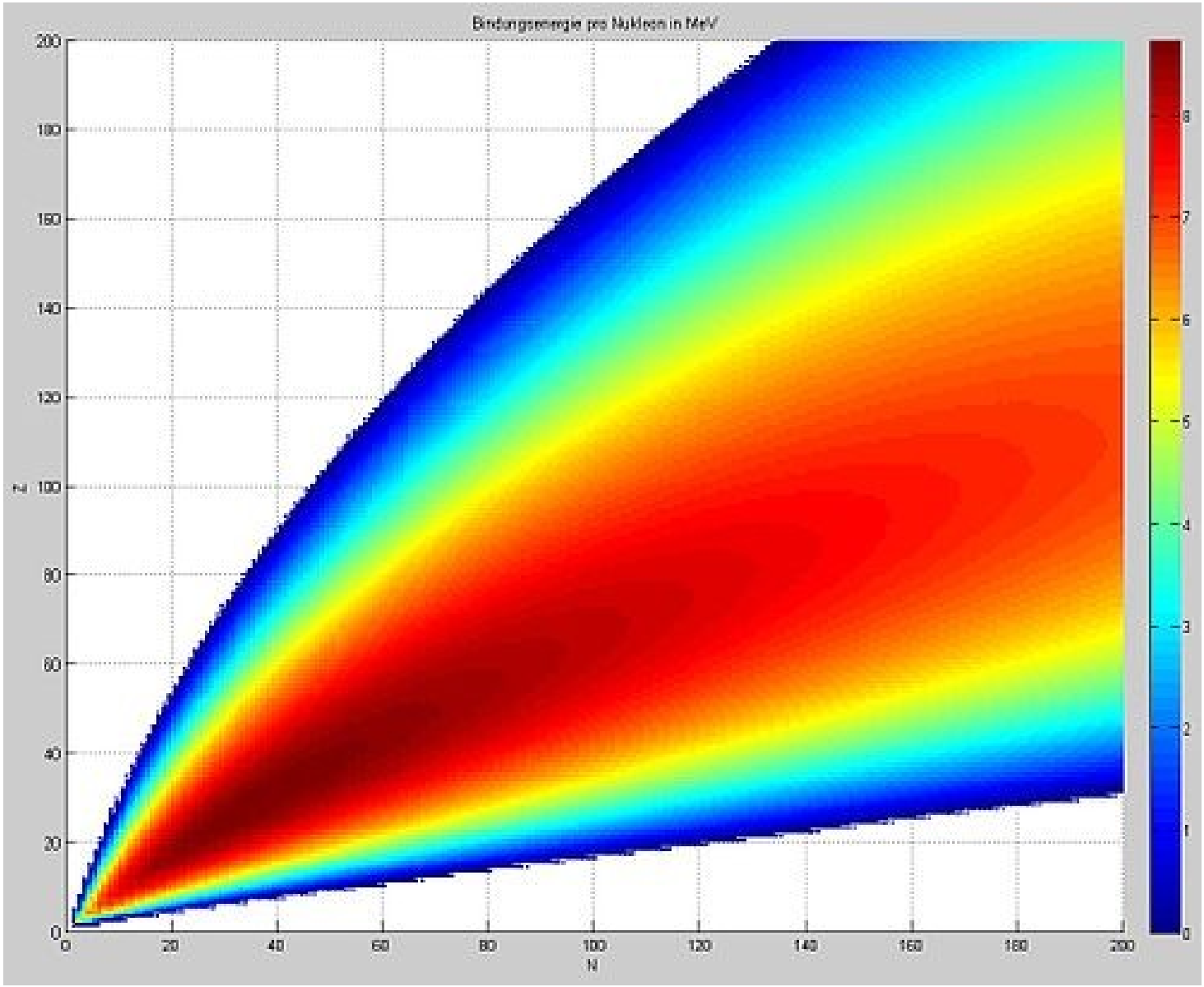
Summary

Energy terms

Best fit to experimental Binding Energy curve yields

$$a_V = 15.5 MeV$$
$$a_S = 16.8\ MeV$$
$$a_C = 0.72\ MeV$$
$$a_{Sym} = 23\ MeV$$
$$a_{pair} = 34\ MeV$$

Liquid drop energy versus Z & N



B.04. Spherical operators

are tensors of rank λ (integer or half-integer)

The main property is given by the **Wigner-Eckart theorem:**
the dependence of the matrix element on spin projections
is given by the Clebsch-Gordan coefficient:

$$\left\langle j_1 m_1 \mid Q_{\lambda\mu} \mid j_2 m_2 \right\rangle = \frac{1}{\sqrt{2j_1+1}} \left\langle j_2 m_2 ; \lambda\mu \mid j_1 m_1 \right\rangle \left\langle j_1 \| Q_\lambda \| j_2 \right\rangle$$

where the last bracket denotes
the reduced matrix element

Notice the inverse order of projections
In the Clebsch-Gordan coefficient!
As an example of a tensor operator
we give the electric multipole operator

$$Q_{\lambda\mu} = r^\lambda Y_{\lambda\mu}(\vartheta, \varphi)$$

B.05. Bohr-Mottelson model (geometrical model)

The nuclear surface is described by surface coordinates as follows:

$$R(\vartheta,\varphi) = R_0\left(1+\sum_{\lambda=1}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\alpha^*_{\lambda\mu}Y_{\lambda\mu}(\vartheta,\varphi)\right)$$

where $\alpha_{\lambda\mu}$ are called surface coordinates in the laboratory system.
Both α and Y are spherical tensors of rank λ.
The above summation on μ is proportional to the angular momentum coupling to 0:

$$\sum_{\mu=-\lambda}^{\lambda}\alpha^*_{\lambda\mu}Y_{\lambda\mu} = \sum_{\mu=-\lambda}^{\lambda}(-)^{\lambda-\mu}\alpha_{\lambda,-\mu}Y_{\lambda\mu} = \sqrt{2\lambda+1}\left(\alpha_\lambda \otimes Y_\lambda\right)_0$$

The most important degrees of freedom are:
-**Quadrupole coordinates (λ=2)**
describing an ellipsoidal shape
-**Octupole coordinates (λ=3)**
describing a pear shape

Surface motion is given by a superposition of ho terms with different multipolarities:

$$\hat{\mathbf{H}} = H_0 + \sum_{\lambda\mu} \hat{\mathbf{H}}_{\lambda\mu}$$

where:
$$\hat{\mathbf{H}}_{\lambda\mu} = \frac{1}{2} B_\lambda \left|\dot{\alpha}_{\lambda\mu}\right| + \frac{1}{2} C_\lambda \left|\alpha_{\lambda\mu}\right|^2$$
$$= \frac{1}{2B_\lambda} \left|\pi_{\lambda\mu}\right|^2 + \frac{1}{2} C_\lambda \left|\alpha_{\lambda\mu}\right|^2$$

with:
$$\pi_{\lambda\mu} = B_\lambda \dot{\alpha}_{\lambda\mu} = -i\hbar \frac{\partial}{\partial \alpha_{\lambda\mu}}$$

Each ho term can be quantized in terms of creation and annihilation phonons:

$$\hat{\boldsymbol{\beta}}^+_{\lambda\mu} = \frac{1}{\sqrt{2}} \left(q_{\lambda\mu} - i p_{\lambda\mu}\right)$$
$$\hat{\boldsymbol{\beta}}_{\lambda\mu} = \frac{1}{\sqrt{2}} \left(q_{\lambda\mu} + i p_{\lambda\mu}\right)$$

where:
$$q_{\lambda\mu} = \frac{\alpha_{\lambda\mu}}{\gamma_\lambda}$$
$$p_{\lambda\mu} = -i\hbar \frac{\partial}{\partial q_{\lambda\mu}}$$
$$\gamma_\lambda \equiv \sqrt{\frac{\hbar}{B_\lambda \omega_\lambda}}; \omega_\lambda = \sqrt{\frac{C_\lambda}{B_\lambda}}$$

Quantized ho Hamiltonian

can be written as follows:

$$\hat{\mathbf{H}} = H_0 + \sum_{\lambda} \left(\hat{\mathbf{N}}_{\lambda} + \frac{1}{2} \right) \hbar\omega_{\lambda}$$

in terms of the number of phonons operator:

$$\hat{\mathbf{N}}_{\lambda} = \sum_{\mu} \hat{\boldsymbol{\beta}}^{+}_{\lambda\mu} \hat{\boldsymbol{\beta}}_{\lambda\mu}$$

The energy is the sum of ho energies:

$$E = E_0 + \sum_{\lambda\mu} \left(n_{\lambda\mu} + \frac{1}{2} \right) \hbar\omega_{\lambda}$$

Vibrational states

one phonon states

- Even-even nuclei
- A vibrating liquid drop.
- Average shape is spherical

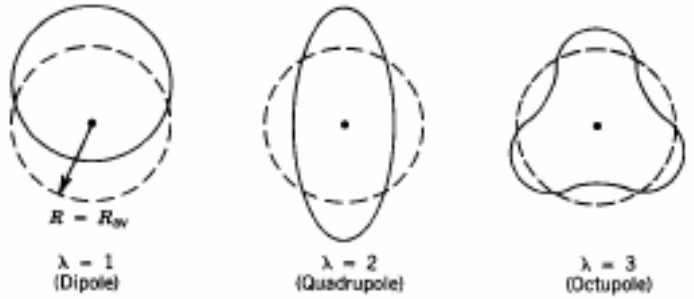


Figure 5.18 The lowest three vibrational modes of a nucleus. The drawings represent a slice through the midplane. The dashed lines show the spherical equilibrium shape and the solid lines show an instantaneous view of the vibrating surface.

Krane 5.18

- Dipole mode violates conservation of momentum.
- Centre of mass is displaced – can't be the result of internal forces.

Two phonon quadrupole states

$$(\hat{\boldsymbol{\beta}}_2^+ \otimes \hat{\boldsymbol{\beta}}_2^+)_J$$

$$J = 0,2,4$$

Quadrupole mode: adds a 2nd-order spherical harmonic to the nuclear wavefunction

2 units of angular momentum (l)

Phonon – a quantized vibration

$$\pi = (-1)^l = +1 \qquad \therefore\ 0^+ \rightarrow 2^+$$

Two phonons result in a triplet

$$\therefore\ 0^+,\ 2^+, 4^+$$

Energy = 2 x Energy of one-phonon 2^+ state

Phonon octupole states

- Octupole mode: adds a 3rd-order spherical harmonic to the nuclear wavefunction
- 3 units of angular momentum (l)

$$\pi = (-1)^3 = -1 \qquad \therefore\ 0^+ \rightarrow 3^-$$

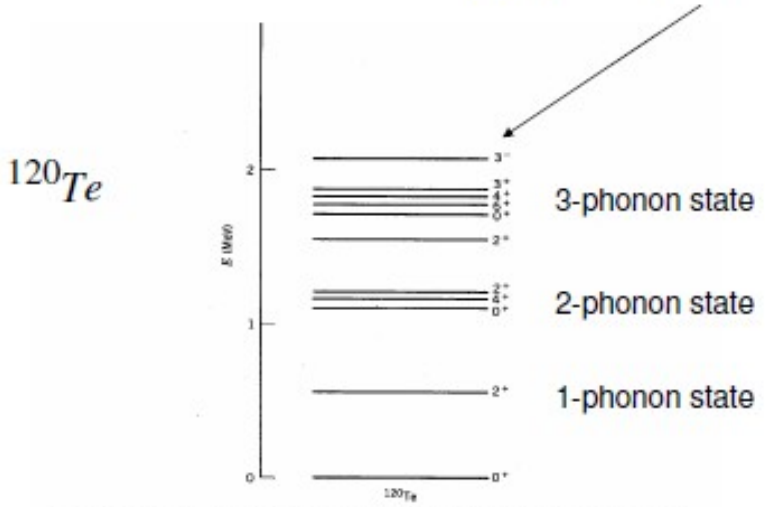
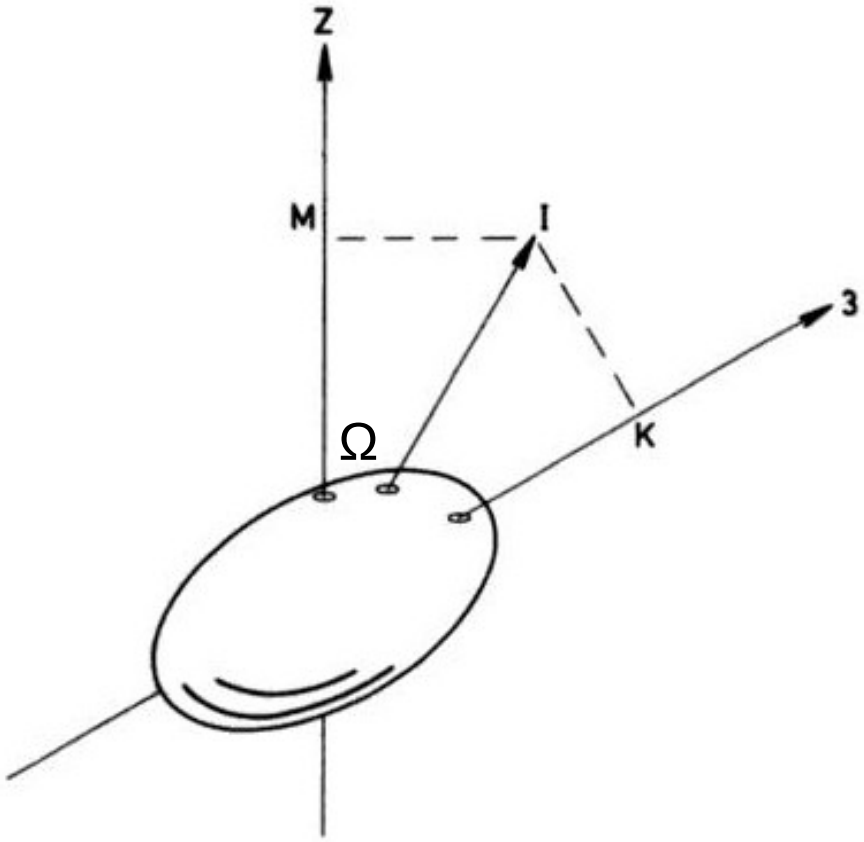


Figure 5.19 The low-lying levels of ^{120}Te. The single quadrupole phonon state (first 2^+), the two-phonon triplet, and the three-phonon quintuplet are obviously seen. The 3^- state presumably is due to the octupole vibration. Above 2 MeV the structure becomes quite complicated, and no vibrational patterns can be seen.

B.06. Intrisic system of coordinates

is connected to the symmetry axis of a deformed nucleus “3”.
Total spin I has projections
in the laboratory (M) and intrinsic (K) systems.
Ω is the rotation angle between the two systems.



Surface coordinates in the intrinsic system

connected to the nuclear symmetry axis “3”
are given by a rotation Ω of $\alpha_{2\mu}$
from the laboratory system “z”
and have the following expressions:

$$a_{20} = \beta \cos\gamma$$

$$a_{22} = \frac{\beta}{\sqrt{2}} \sin\gamma$$

where
β describes axial deformation
γ describes triaxial deformation

Quadrupole kinetic energy

in the intrinsic system can be written as follows:

$$T_2 = \frac{1}{2} B_2 \left(\dot{\beta}^2 + \beta^2 \dot{\gamma}^2 \right) + \frac{1}{2} \sum_{k=1}^{3} I_k \omega_k^2$$

where the moments of inertia are:

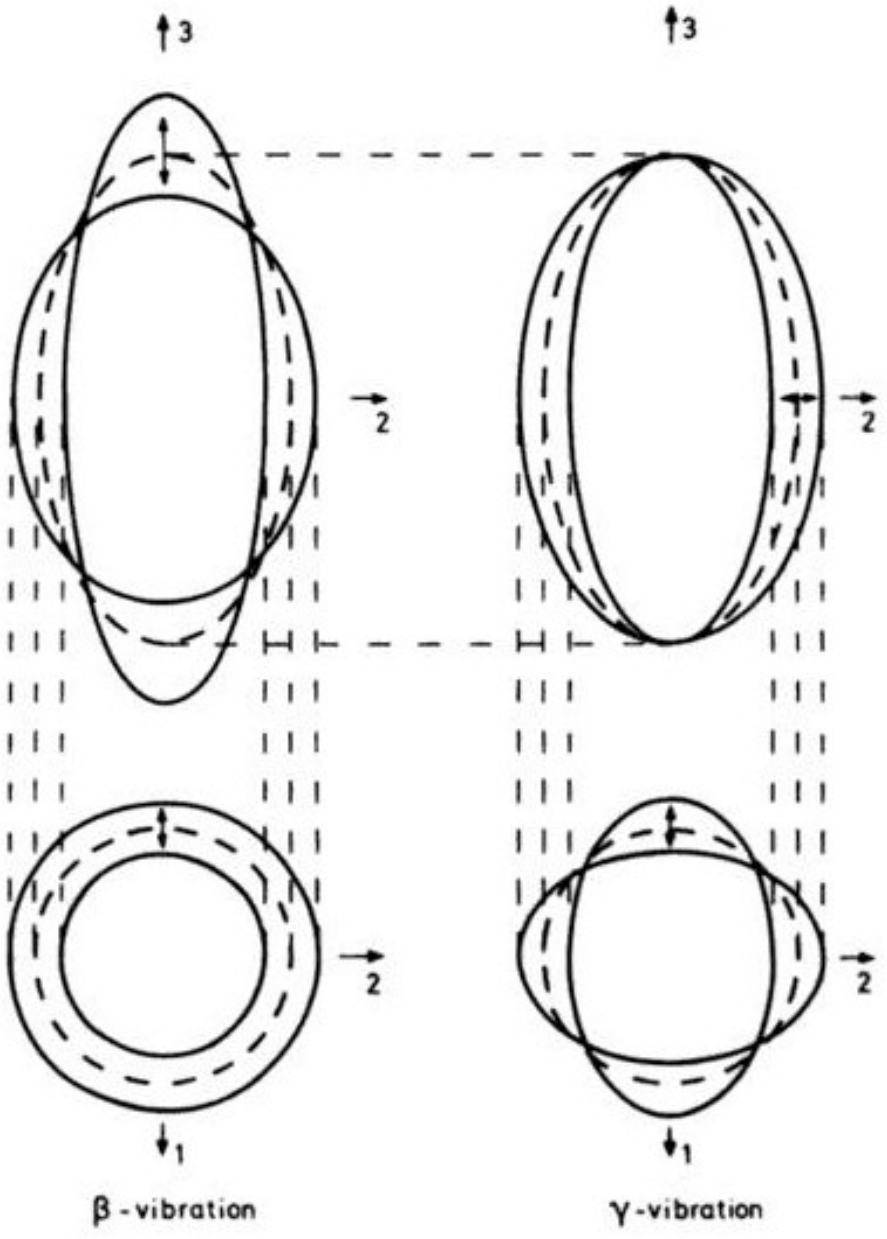
$$I_k = 4\beta^2 \sin^2\left(\gamma - \frac{2\pi}{3} k \right)$$

The second term gives the rotational energy and ω_k are rotation frequencies.

Most of nuclei are axially symmetric with *γ=0*.
They have vibrational and rotational energy:

$$T_2 = \frac{1}{2} B_2 \dot{\beta}^2 + \frac{1}{2} I_3 \omega_3^2$$

β & γ vibrations of a deformed shape



Rotational states for axially symmetric nuclei (γ=0)

Rotational states are observed in deformed nuclei within the range A=150 to 190 and A>220.

The rotational energy is:

$$E_{rot} = \frac{1}{2} I\omega^2 = \frac{L^2}{2I}$$

where the momentum of inertia is defined from:

$$L = I\omega$$

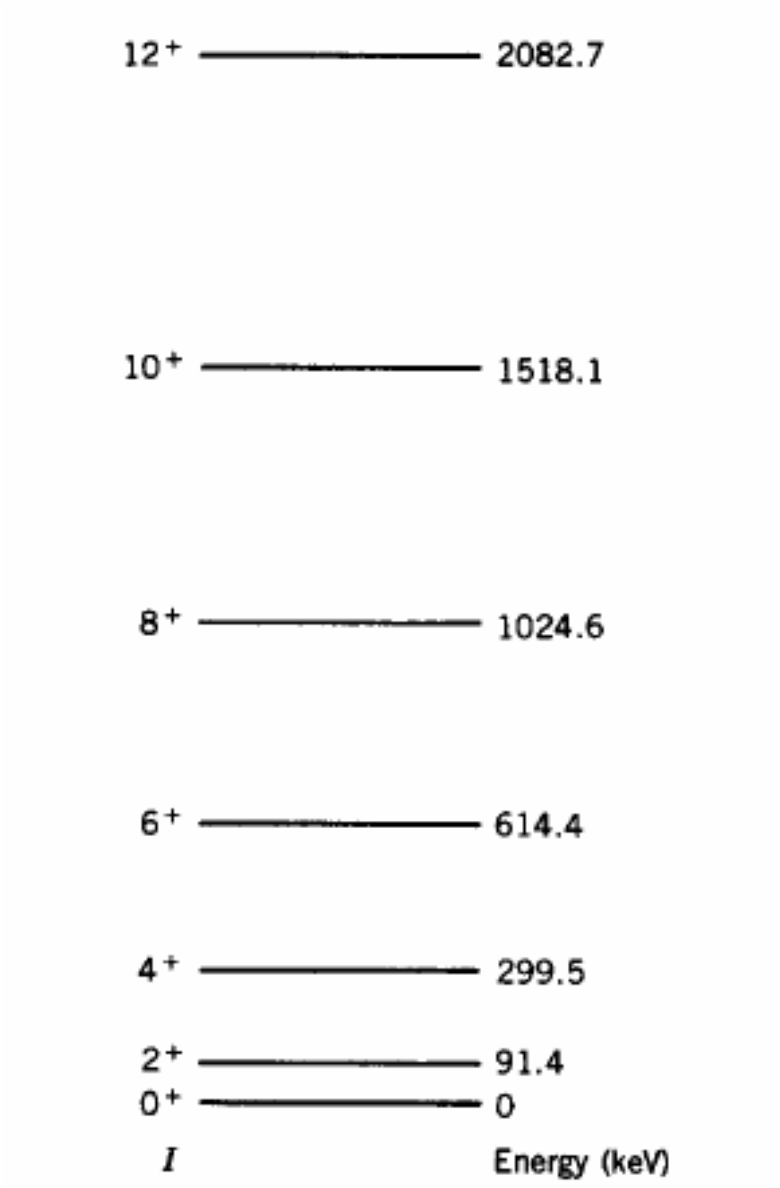
By using quantum expression of the angular momentum

$$L \rightarrow \hbar^2 l(l+1)$$

one obtains rotational energy:

$$E_{rot} = \frac{\hbar^2 l(l+1)}{2I}$$

Rotational states in ^{164}Er



Rotational band
has the energy:

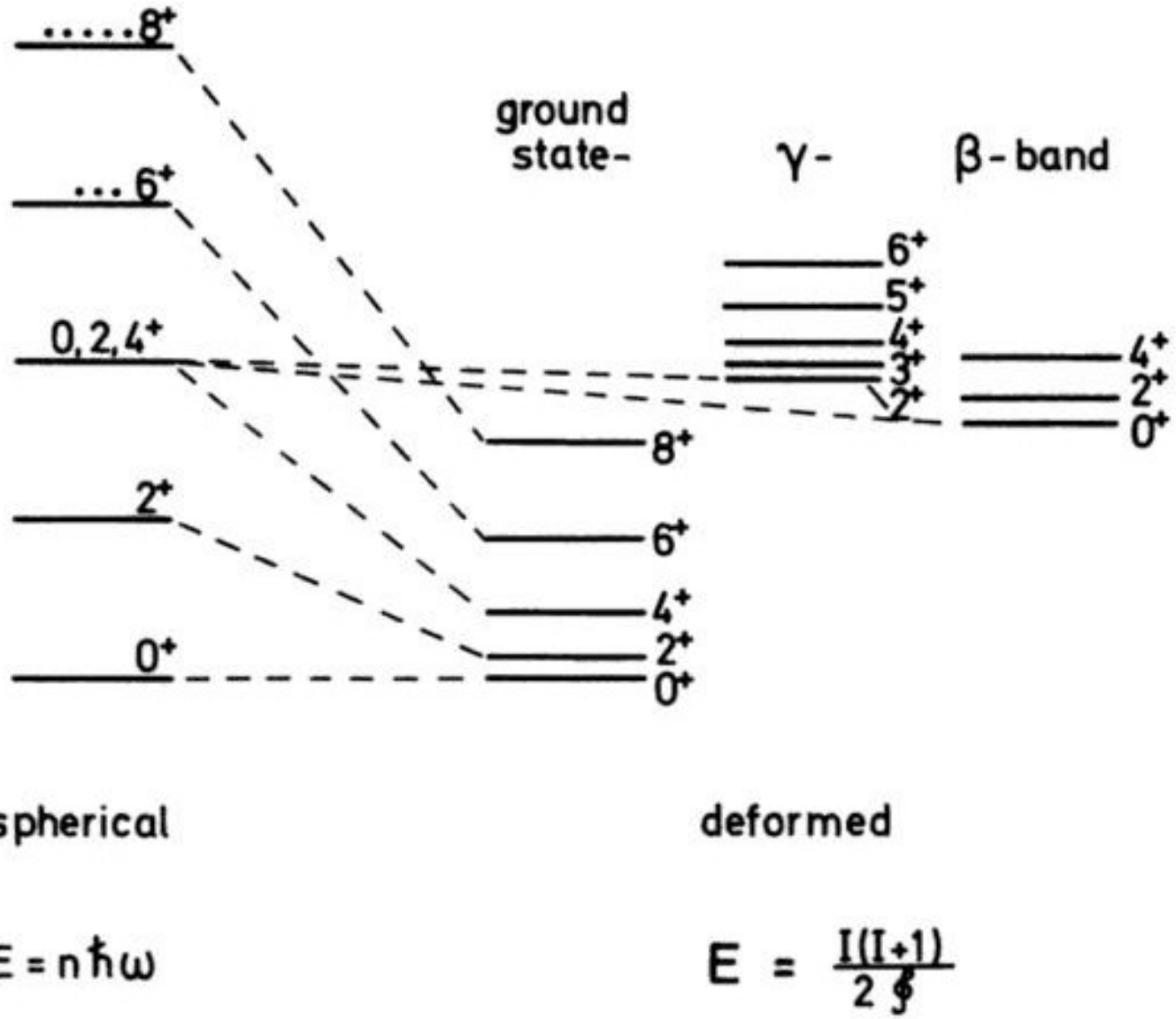
$$E_{Kn_\beta n_\gamma}(J) = E_{Kn_\beta n_\gamma}(0) + \frac{\hbar^2}{2I_0}\left[J(J+1) - K^2\right]$$

and it is built on top of the vibrational band head
with energy given by β and γ vibrations:

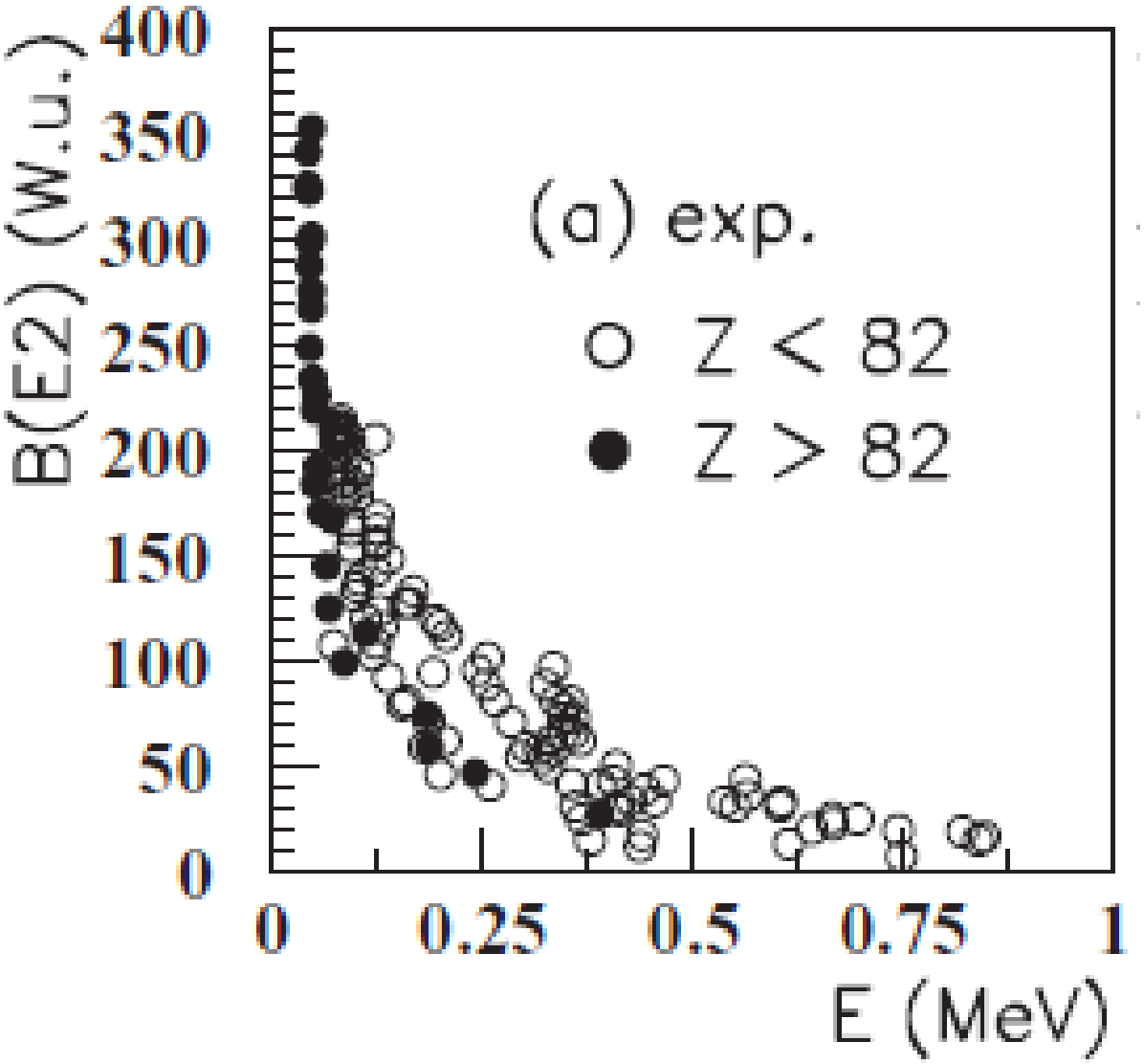
$$E_{Kn_\beta n_\gamma}(0) = \hbar\omega_\beta\left(n_\beta + \frac{1}{2}\right) + \hbar\omega_\gamma\left(2n_\lambda + 1 + \frac{|K|}{2}\right)$$

Sakai-Sheline rule

describes how vibrational states → rotational bands
when the quadrupole deformation increases



Electric quadrupole transitions between low-lying states
of the same band are very strong, proportional to mass number
and inverse proportional to the excitation energy (in W.u.).
Below we give experimental values B(E2:2^+->0^+) as a function of the energy E_2:



B.07. Interacting Boson Model (IBM) is an algebraic model (optional lecture)

IBM 6-dimensional basis is built from monopole bosons “s” and quadrupole bosons “d”:

$$b^+_\mu = \begin{pmatrix} s^+ \\ d^+_{-2} \\ ..... \\ d^+_2 \end{pmatrix}$$

36 bilinear generators

$$\hat{\mathbf{G}}_{\mu\nu} = b^+_\mu b_\nu \qquad \mu,\nu = 1,..,6$$

form the U(6) algebra (see next page)

The most general Hamiltonian written in terms of these generators

$$\hat{\mathbf{H}} = \sum_{\mu\nu} \varepsilon_{\mu\nu} \hat{\mathbf{G}}_{\mu\nu} + \frac{1}{2} \sum_{\mu\nu\alpha\beta} U_{\mu\nu\alpha\beta} \hat{\mathbf{G}}_{\mu\nu} \hat{\mathbf{G}}_{\alpha\beta} = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2$$

should be invariant with respect to various transformations which form sub-algebra of U(6).

Definitions

Algebra (in a restricted sense) is a space built on a set of independent operators called generators, such that the commutator between generators is a linear combination of them.

Relation between the group and its algebra

For instance O(3) algebra is built on components of the angular momentum L_x, L_y, L_z, which are the generators of the rotation group.

Rotation group is represented by an operator depending on the rotation angle and acting on the space of functions.

For the rotation around z axis the parameter is the rotation angle φ:

$$\hat{\mathbf{R}}(\varphi) f(\alpha) = f(\alpha - \varphi) = e^{-\varphi \frac{\partial}{\partial \varphi}} f(\alpha)$$

group → $\hat{\mathbf{R}}(\varphi) \rightarrow e^{-\varphi \frac{\partial}{\partial \varphi}} = e^{-\frac{i}{\hbar} \varphi \hat{L}_z}$ ← algebra

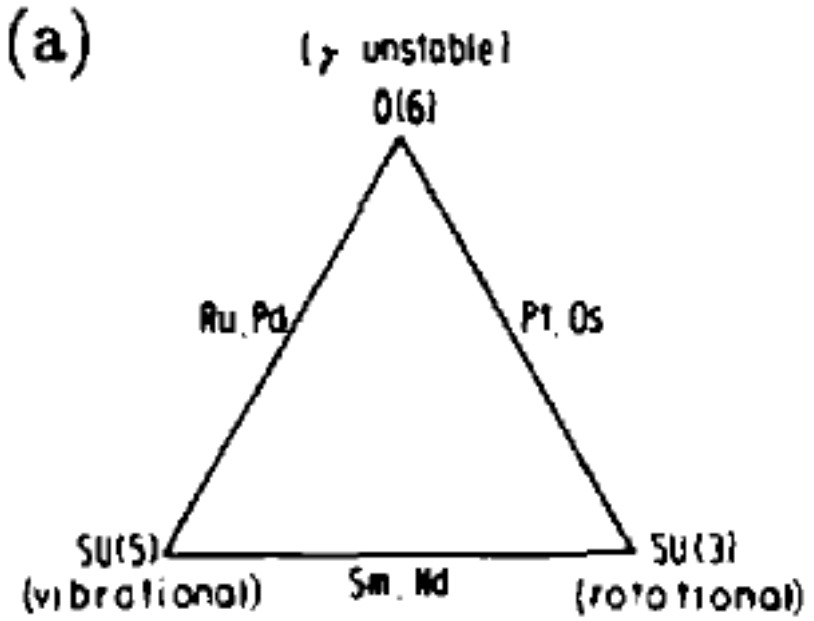
Examples

U(n) unitary group: n x n unitary matrices

SU(n) special unitary group: U(n) with determinant 1

O(n) orthogonal group: conserves the Euclidean distance

The invariants of various algebra are called **Casimir operators**.
The part $\mathbf{H_2}$ of the Hamiltonian can be written as a sum of different Casimir operators, expressing **dynamical symmetries**.
The main sub-algebra of U(6) are
SU(3) : rotational limit,
SU(5): vibrational limit and
O(6): gamma soft (independent on angle γ).
Hamiltonian containing only one of the Casimir operators correspond to a corner of the **Casten triangle**



IBM-1 describes collective excitations which do not depend on isospin.
IBM-2 describes proton and neutron collective excitations.

The most popular IBM-2 Hamiltonian is given by:

$$\hat{\mathbf{H}} = \sum_{\tau=\pi,\nu} \varepsilon_\tau s_\tau^+ . s_\tau + \sum_{\tau=\pi,\nu} \varepsilon_\tau d_\tau^+ . d_\tau - \sum_{\tau\tau'=\pi,\nu} \kappa_{\tau\tau'} Q_\tau . Q_{\tau'}$$

$$Q_{\tau\mu} = d_{\tau\mu}^+ s_\tau + s_\tau^+ \tilde{d}_{\tau\mu} + \chi_\tau \left(d_\tau^+ \otimes \tilde{d}_\tau \right)_{2\mu}$$

$$\tilde{d}_{\tau\mu} = (-)^\mu d_{\tau-\mu}$$

B.08. Giant resonance

is the motion of proton and neutron systems with opposite phases, excited by an external electromagnetic field

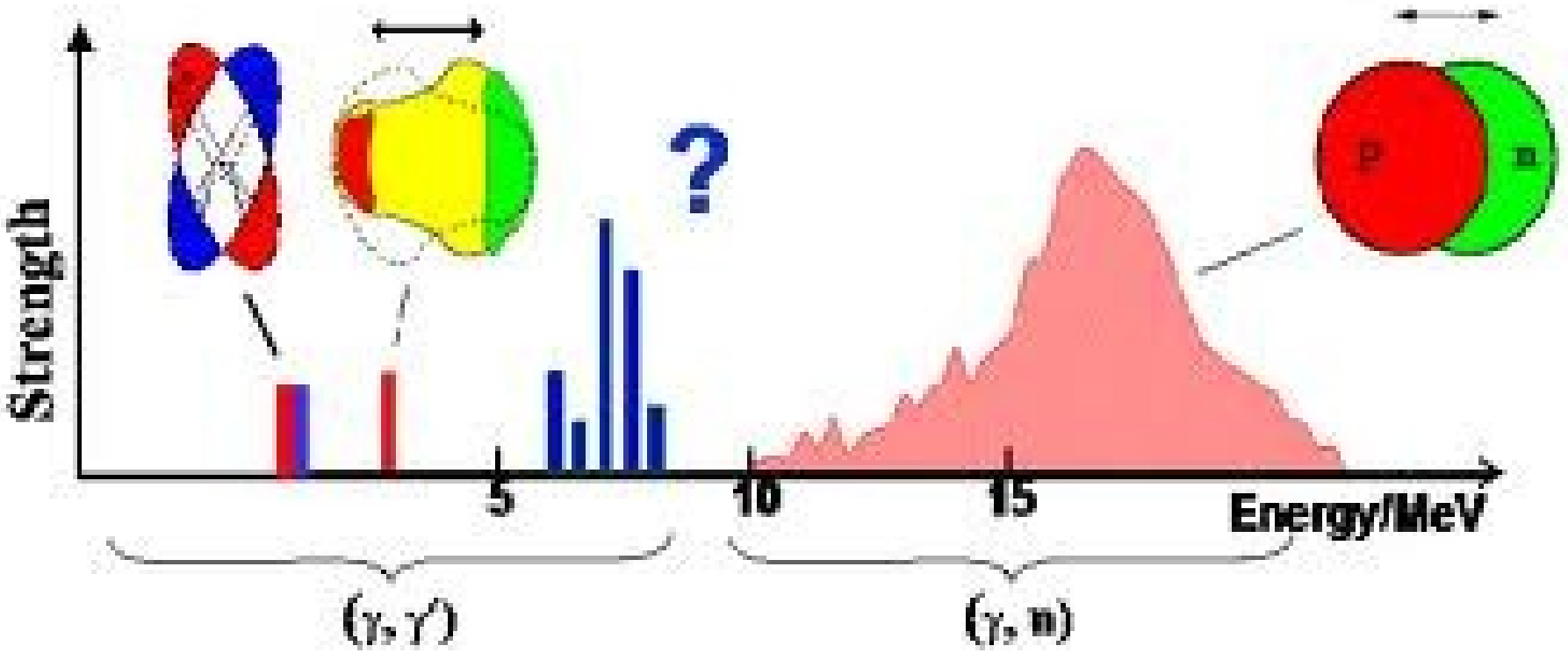
Golhaber-Teller model assumes a quadratic dependence of the energy between proton and neutron systems upon the radius *r* between their distributions:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}kr^2 \qquad \mu = \frac{ZNm}{A}$$

k is estimated from the symmetry energy, expressing binding energy versus $(N-Z)^2$
The energy of the harmonic oscillator is:

$$\hbar\omega = \hbar\sqrt{\frac{3a_{sym}}{4\varepsilon m}}\sqrt{\frac{A^2}{ZNR}} \approx \frac{45}{A^{1/6}}\,MeV$$

M1 (magnetic dipole: “scissors mode”) , E3 (electric octupole) and E1 (electric dipole) giant resonances



? denotes “Pygmy mode” or “surface resonance”, where only the “skin neutrons” on the nuclear surface oscillate

Isoscalar (IS), isovector (IV) giant (G) or surface (S) monopole (M), dipole (D), quadrupole (Q) resonaces (R)