Scientific report for the period september 2013-december 2016

Project Title : Topics in non-abelian gauge theories

Project Director: Catalina Renata Jora

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I. SCIENTIFIC REPORT SEPTEMBER 2013-DECEMBER 2016

In the period september 2013-december 2016 I sent for publication twenty four papers: six were published in Int. J. Mod. Phys. A, two in Mod. Phys. Lett. A, one in Phys. Lett. B, seven were accepted or published in Rom. J. Phys., Rom. Rep. Phys., Rom. Acad. J. and eight are in the process of evaluation at different journals.

The program for the year 2013 contained one main objective:

1) A new effective symmetry of the gauge theory and connection with confinement. This objective included two activities:

a) Introduction of a new effective symmetry of the gauge theories.

b) Applications of this new symmetry.

The program for the year 2014 contained three main objectives:

1) Partition function for a scalar theory without spontaneous symmetry breaking using a new method of functional integration. This objective comprised two activities:

a) Introduction of a new method of functional integration.

b) The scalar propagator in this approach.

2) Partition function for a gauge theory with fermions in a new functional approach. This objective included two activities:

a) Introduction of a new functional approach for a theory with fermions.

b) Correlation functions in this approach and the beta function for the electric charge.

3) Partition function for a gauge theory with strong interaction in a new functional approach. This objective comprised one activity:

a) Application to gauge theories with strong interaction.

The program for the year 2015 encompassed three objectives:

1) Higher orders or all orders beta function for nonabelian gauge theories with fermions. This objective included one activity:

a) An analytical evaluation of the higher orders or all orders beta function for a nonabelian gauge theory with fermions.

2) A new perspective on the phase transitions for gauge theories. This objective comprised one activity:

a) Applications of the functional formalism to the study of phase transitions.

3) Phase transitions for nonabelian gauge theories. This objective contained one activity:

a) A study of the phase transitions for nonabelian gauge theories.

The program for the year 2016 included two objectives:

1) Partition function for a scalar theory with spontaneous symmetry breaking. This objective encompassed two activities:

a) Partition function for a theory with spontaneous symmetry breaking in the new functional approach.

b) Corrections to the scalar field propagator and mass in the new method.

2) Applications of the new method to the standard model of elementary particles. This objective comprised two activities:

a) Partition function in the new approach.

b) Corrections to the Higgs boson mass or information pertaining to it in our new method.

We shall discuss the scientific results in accordance with the objectives.

A. A new effective symmetry of the gauge theories and connection with confinement.

The standard model of elementary particles has become one of the most successful theories of modern physics with enumerable experimental confirmations culminating with the discovery of a Higgs like particle at the LHC . However there are still theoretical issues to be clarified some of them in connection with the cosmological issues such as the origin of matter-antimatter asymmetry in the universe and the candidate for the dark matter. There are many established extensions of the standard model that solve some of these problems like the supersymmetric standard model, technicolor models, two Higgs doublet models, GUT models etc. Among these only the experiment can decide. The present work starts with two fundamental question apparently unrelated: a) what is the role of the scalars in a gauge theory with fermions; b) why are the neutrino masses so small compare to the other fermions such that in first order approximation may be taken as zero? We will show here that a particular answer to these two problems can be given in the context of a new symmetry of the standard model Lagrangian.

Consider an abelian $U(1)$ theory with massless fermions:

$$\mathcal{L} = \bar{\Psi}(i\gamma^{\mu}D_{\mu})\Psi - \frac{1}{4}(F_{\mu\nu})^2, \tag{1.1}$$

where,

$$D_\mu = \partial_\mu - igA_\mu. \tag{1.2}$$

This Lagrangian not only preserves the Lorentz and gauge invariance but also the chirality of the states. It turns out that this theory is also invariant under the infinitesimal transformation given by the operator $K = \exp[k\gamma^\mu D_\mu]$ which is non-unitary and unbounded (and does not satisfy the premises of the Coleman Mandula theorem):

$$\begin{aligned} \Psi' &= \Psi + k\gamma^\mu D_\mu \Psi \\ A'_\mu &= A_\mu \end{aligned} \tag{1.3}$$

where k is an inverse scale such that for a square momentum p^2, $p^2k^2 \ll 1$. This is proven by:

$$\begin{aligned} & i\bar{\Psi}'\gamma^\mu D_\mu \Psi' = i\bar{\Psi}\gamma^\mu D_\mu \psi + \\ & ik(D_\rho\Psi)^\dagger \gamma^{\rho *}\gamma^0\gamma^\mu D_\mu\Psi + ik\Psi^\dagger\gamma^0\gamma^\mu\gamma^\rho D_\mu D_\rho\Psi = \\ & i\bar{\Psi}\gamma^\mu D_\mu\Psi - ik\Psi^\dagger\gamma^0\gamma^\rho\gamma^\mu D_\rho D_\mu\Psi + ik\Psi^\dagger\gamma^0\gamma^\mu\gamma^\rho D_\mu D_\rho\Psi = \\ & i\bar{\Psi}\gamma^\mu D_\mu\Psi \end{aligned} \tag{1.4}$$

We generalize the Lagrangian in Eq. (1.1) to contain also a neutral scalar which couples with the fermion fields:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}m^2B^2 + \bar{\Psi}(i\gamma^\mu D_\mu)\Psi - m_f\bar{\Psi}\Psi - yB\bar{\Psi}\Psi - \frac{1}{4}(F_{\mu\nu})^2. \tag{1.5}$$

Here B is the scalar field and y is the Yukawa coupling for the fermion field. Applying the infinitesimal transformation given by the operator K to the Yukawa term yields:

$$\begin{aligned} -\Psi'^\dagger\gamma^0 B\Psi' &= -(K\Psi)^\dagger\gamma^0 B(K\Psi) \\ &= -\Psi^\dagger\gamma^0 B\Psi - k(D_\mu\Psi)^\dagger\gamma^{*\mu}\gamma^0 B\Psi - k\Psi^\dagger\gamma^0\gamma^\mu BD_\mu\Psi \\ &= -\Psi^\dagger\gamma^0 B\Psi + k\Psi^\dagger\gamma^0\gamma^\mu BD_\mu\Psi - k\Psi^\dagger\gamma^0\gamma^\mu BD_\mu\Psi + k\Psi^\dagger\gamma^0\gamma^\mu\partial_\mu B\Psi \\ &= -\Psi^\dagger\gamma^0 B\Psi + k\Psi^\dagger\gamma^0\gamma^\mu\partial_\mu B\Psi. \end{aligned} \tag{1.6}$$

It turns out that this term is not invariant under the symmetry because of the extra term which contains a partial derivative of B. However the Lagrangian in Eq. (1.5) is invariant under a slightly modified symmetry given by the operator $K' = \exp[k\gamma^\mu D'_\mu]$, which for a infinitesimal transformation is defined as:

$$\begin{aligned}
\Psi' &= \Psi + k\gamma^\mu D'_\mu \Psi \\
A'_\mu &= A_\mu - \frac{k}{g}\partial_\mu B \\
B' &= B
\end{aligned} \tag{1.7}$$

where,

$$D'_\mu = \partial_\mu - igA'_\mu. \tag{1.8}$$

The invariance of the Lagrangian (1.5) under the symmetry defined in (1.7) can be proven as follows (for simplicity we set the Yukawa coupling $y = 1$):

$$\begin{aligned}
\mathcal{L}' &= i\Psi^\dagger\gamma^0\gamma^\mu D'_\mu\Psi - \Psi^\dagger\gamma^0 B\Psi + ik(\gamma^\rho D'_\rho\Psi)^\dagger\gamma^0\gamma^\mu D'_\mu\Psi \\
&\quad + ik\Psi^\dagger\gamma^0\gamma^\mu D'_\mu\gamma^\rho D'_\rho\Psi - k(\gamma^\rho D'_\rho\Psi)^\dagger\gamma^0 B\Psi - k\Psi^\dagger\gamma^0\gamma^\rho D'_\rho\Psi \\
&= i\Psi^\dagger\gamma^0\gamma^\mu D'_\mu\Psi + k\Psi^\dagger\gamma^0\gamma^\mu\partial_\mu B\Psi - \Psi^\dagger\gamma^0 B\Psi \\
&= \mathcal{L} + g\Psi^\dagger\gamma^0\gamma^\mu(A'_\mu - A_\mu)\Psi + k\Psi^\dagger\gamma^0\gamma^\mu\partial_\mu B\Psi.
\end{aligned} \tag{1.9}$$

where we have taken into account only the relevant terms. Note that the symmetry K' in (1.7) identifies the gauge parameter α up to a proportionality factor with the dynamical scalar in the model B, i.e. $\alpha = kB$.

For the K' to be a true symmetry of a gauge theory we need to prove that it is valid also for a non-abelian gauge theory. This is shown in,

$$\begin{aligned}
-\bar{\Psi}' B_1 \Psi' &= -\bar{\Psi} B_1 \Psi - k(\gamma^\rho D'_\rho\Psi)^\dagger\gamma^0 B_1\Psi - k\Psi^\dagger\gamma^0\gamma^\rho B_1 D'_\rho\Psi = \\
&= k\Psi^\dagger\gamma^0\gamma^\rho(D_\rho B_1)\Psi.
\end{aligned} \tag{1.10}$$

Here B_1 is a scalar field which may be elementary or composite and for the case of QCD for example has the structure $B_1 = B_1^a\frac{\lambda_a}{2}$. Thus a non-abelian gauge Lagrangian is invariant provided that,

$$g(A'_\mu - A_\mu) = -kD_\mu B_1, \tag{1.11}$$

So we again obtain the relation $\alpha = kB_1$ but this time for a non-abelian gauge parameter (for the case of QCD $\alpha = \alpha^a\frac{\lambda^a}{2}$). Hence we showed that the standard model of elementary particles is invariant under the infinitesimal symmetry given by the operator K' provided that his operator is adjusted to each fermion species and contains the covariant derivative or simple derivatives corresponding to the each species kinetic term. For example the quarks transformations are given by:

$$q_i' = q_i + k_i \gamma^\mu D_\mu q_i, \tag{1.12}$$

where i represents the quark flavor, $D_\mu = (\partial_\mu - ieA_\mu - igA_\mu^a \frac{\lambda_a}{2})$ is the full covariant derivative which contains the electromagnetic and the color gauge fields A_μ and A_μ^a, respectively..

To complete the proof that K' is a symmetry of the standard model Lagrangian we need to consider the transformation of the charged and neutral currents interaction. They have the form:

$$\begin{aligned} \mathcal{L}_{CC} &= \frac{g}{\sqrt{2}}(J_\mu^+ W_\mu^+ + h.c.) \\ \mathcal{L}_{NC} &= \frac{g}{\cos\theta_W} J_\mu^0 Z^\mu \end{aligned} \tag{1.13}$$

where the currents $J_\mu^{\pm,0}$ have a structure of the type $\bar{f}_L^1 \gamma^\mu \bar{f}_L^2$ for the charged currents or $\bar{f}_{L,R}^1 \gamma^\mu \bar{f}_{L,R}^1$for the neutral ones. We shall consider only the first case. Under K', a left handed fermion transforms as

$$K'\Psi_L = \Psi_L + k\gamma^\mu D_\mu'(\frac{1-\gamma_5}{2})\Psi = \Psi_L + \frac{1+\gamma^5}{2}\gamma^\mu D_\mu' \Psi. \tag{1.14}$$

Consequently, the action of this operator on the charged current Lagrangian leads to,

$$\begin{aligned} \mathcal{L}_{CC}' &= \frac{g}{2}(K'f^1)^\dagger \gamma^0 \gamma^\mu (K' f_2) W_\mu^+ + h.c.) \frac{g}{2}(f^1)^\dagger \gamma^0 \gamma^\mu f^2 W_\mu^+ + \\ &= k_1 \frac{g}{2}(D_\rho' f^1)^\dagger \gamma^{\rho *}(\frac{1+\gamma^5}{2})\gamma^0\gamma^\mu(\frac{1-\gamma^5}{2}) f^2 W_\mu^+ + \\ &+ k_2 \frac{g}{2} f^{1\dagger}(\frac{1-\gamma^5}{2})\gamma^0\gamma^\mu(\frac{1+\gamma^5}{2})\gamma^\rho D_\rho' f^2 W_\mu^+ + h.c. \\ &= \frac{g}{2}(f^{1\dagger}\gamma^0\gamma^\mu f^2 W_\mu^+ + h.c.) = \mathcal{L}_{CC}. \end{aligned} \tag{1.15}$$

This concludes the proof that the operator K' , which contains all the unbroken gauge interactions specific to each fermion, is a symmetry of the standard model Lagrangian invariant .

In the previous paragraphs we have shown that the symmetry given by the operator $K' = \exp[k\gamma^\mu D_\mu']$ works as an infinitesimal transformation acting on the fermion fields in a gauge invariant Lagrangian. Thus we have made the underlying assumption that the constant $k = \frac{1}{\Lambda}$ functions as a natural cut-off of the theory such that all the momenta are lower than this scale. We know that the gauge symmetry is a good symmetry both as infinitesimal transformation and finite one. It is natural to inquire what happens if the inverse scale k is of the size of the fermion momentum of one particular species of fermions. Since we expect the nonabelian case to be more interesting we shall consider the QCD Lagrangian with only one flavor and with an additional quark-quark-scalar interaction, where the scalar may be elementary or composite:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}G_{\mu\nu}G^{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - y\bar{\Psi}B_1\Psi. \tag{1.16}$$

Previously we have proven the invariance of this Lagrangian under the transformation:

$$\begin{aligned} \Psi' &= \Psi + k\gamma^\mu D'_\mu \Psi \\ A'_\mu &= A_\mu - \frac{k}{g} D_\mu B_1 \\ B'_1 &= B_1. \end{aligned} \tag{1.17}$$

We expect that the requirement that the Lagrangian in Eq. (1.16) be invariant under the full symmetry K' will lead to particular solutions for the fields involved. In what follows we show that this is indeed the case. So we require that

$$(K'\Psi^\dagger)\gamma^0(i\gamma^\mu D'_\mu - m - B_1)(K'\Psi) = \Psi^\dagger\gamma^0(i\gamma^\mu D_\mu - m - B_1)\Psi. \tag{1.18}$$

The most obvious solution to the above equation is then:

$$\begin{aligned} \ln(K')\Psi &= i\alpha\Psi \\ (-igk\gamma^\mu(A'_\mu - A_\mu) + ikm + ikB_1)\Psi &= i\alpha\Psi, \end{aligned} \tag{1.19}$$

where we used the equation of motion for the fermion field.

Now we reinforce $kB_1 = \alpha$, and $g(A'_\mu - A_\mu) = -D_\mu\alpha$, to obtain:

$$k\gamma^\mu D_\mu B_1 \Psi = m\Psi. \tag{1.20}$$

Note that the equation in (1.20) is a constraint for both the fermion and the scalar fields. However we want to extract if possible only the solutions for the scalar field. First we apply the operators in Eq. (1.20) to the field $\bar{\Psi}$ and then multiply to the left to obtain:

$$\bar{\Psi}k^2\gamma^\mu\gamma^\nu(D_\mu B_1)^\dagger(D_\nu B_1)\Psi = m^2\bar{\Psi}\Psi \tag{1.21}$$

Then we extract the trace condition with the append that the final solution we shall obtain from this satisfies also the full Eq. (1.21):

$$\mathrm{Tr}[k^2\gamma^\mu\gamma^\nu(D_\mu B_1)^\dagger(D_\nu B_1)] = 4m^2 \tag{1.22}$$

This equation works as a constraint to the Klein Gordon equation in both cases where the field is elementary or as the equation of motion if the field is composite. We simplify Eq. (1.22) to:

$$\partial^\mu B_1^a \partial_\mu B_1^a + g^2 f^{abc} A^{\mu b} B_1^c f^{amn} A_\mu^m B_1^n = \frac{m^2}{k^2}. \tag{1.23}$$

One of the most interesting solutions to Eq. (1.23) is:

$$B_1 = \frac{m}{k}(x_\mu - x_{0\mu})c^\mu \sum_a \frac{\lambda^a}{2}, \tag{1.24}$$

with $c_0 = 0$, $c_i = 1$ and $A_\mu = 0$. Thus the field B_1 is an effective extended field with constant energy for which a suitable propagator is simply the function $\frac{m^2}{k^2}\delta(\vec{p})'$. An extra factor of $\delta(E - E_0)$ (where E_0 is the constant energy of the scalar field) which fixes the value of the energy should be introduced in the amplitudes. This insures the correct dimension of the corresponding propagator.

We work in the nonrelativistic limit where the four momentum of a fermion is given by $(m, \vec{p})$ such that this type of solution make sense. Then considering the propagator in space time, it is obvious that the type of solution given in Eq. (1.24) applied between two pairs of fermion fields one placed at x_0 the other one at x gives a potential linearly growing with distance of confining type. This potential is attractive as it comes with a positive sign in the interaction hamiltonian.

To see things more clearly let us estimate the potential for the scattering of four fermions intermediated by the field B_1. We can just simply replace in typical Yukawa calculations the propagator by $\frac{m^2}{k^2}\delta(\vec{p})'$ to obtain:

$$V(q) = -i\frac{y^2m^2}{k^2}\delta(\vec{q})' \tag{1.25}$$

which in position space gives

$$V(r) = \int(-i\frac{y^2m^2}{k^2})\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\vec{x}}\delta(\vec{q})' = -\frac{y^2m^2}{k^2}r \tag{1.26}$$

Note that analogously to the Wilson loop, the field B correspond to the gauge parameter of the non abelian gauge theory but the confining behavior is obtained quite straightforwardly by simply reinforcing the gauge symmetry.

Some of the issues associated with the standard model after spontaneous symmetry breaking can be solved if one considers an effective symmetry (the K' symmetry) acting on the fermions in the Lagrangian. This symmetry has some similarities with the BRST symmetry but also some important differences. First of all it is an effective symmetry that depends on the natural cut-off scale in the theory; secondly it identifies the gauge parameters with the scalars, elementary or composite that exist in the theory. However as opposed to the BRST symmetry the scalars are dynamical and present in the theory and one of them can correspond to the Higgs boson found at the LHC. Moreover this simple symmetry can explain why neutrinos which do not have any vector gauge interactions are not coupled at tree level with any scalars (e.g. an $SU(2)_L$ scalar triplet for the case of Majorana neutrinos) : such a coupling would break the K symmetry.

In any gauge theory one needs a gauge condition necessary to eliminate the redundant degrees of freedom. We did that (for the U(1) gauge group) in the context of the K symmetry by fixing regularly the gauge field condition which produces an extra constraint related to the scalar field. Thus its mass is fixed by the theory and the off-shell degrees of freedom are eliminated. This might create some problems with unitarity of the model for the case of a single scalar. However this new symmetry is better realized in the presence of multiple scalars, case in which some of the scalars survive off-shell.

We have also showed that the scalar field which is proportional to the gauge parameter for the group $SU(3)_C$ can play the role of the Wilson loop for describing confining. Note that this scalar may be elementary which contradicts the observation, or composite which better fits the general picture of QCD. In the latter case the scalar is introduced in theory as an auxiliary field which is eliminated by the equation of motion, thus leading to four fermion interactions.

B. Partition function for a scalar theory without spontaneous symmetry breaking using a new method of functional integration.

Currently very much is known about the perturbative behavior of many theories with or without gauge fields. Beta functions for the Φ^4 theory and QED is known up to the fifth order whereas for QCD is known up to the fourth order . However there is limited knowledge regarding the non-perturbative behavior of the same theories. Recently attempts have been made for determining the existence in some renormalization scheme of all order beta functions for gauge theories with various representations of fermions. It is rather useful to search for alternative methods which may reveal either the higher orders of perturbation theories or even the non-perturbative regime.

Here we shall consider the massive Φ^4 theory as a laboratory for implementing a method that can be further applied to more comprehensive models. There is an ongoing debate with regard to the behavior of the renormalized coupling λ at small momenta referred to as "the triviality problem". With the hope that our approach might shed light even on this problem we introduce a new variable in the path integral formalism which allows for a more tractable functional integration and series expansion. Then we compute in this new method the corrections to the mass of the scalar in all order of perturbation theory. This approach should be regarded as an alternate renormalization procedure. Since the corresponding mass anomalous dimension $\gamma(m^2) = \frac{d\ln m^2}{d\mu^2}$ has the first order (one loop) coefficient universal we verify that the first order correction is correct. However we expect that the next orders are different.

We shall illustrate our approach for a simple scalar theory, given by the Lagrangian:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_1 \\
\mathcal{L}_0 &= \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{1}{2}m_0^2\Phi^2 \\
\mathcal{L}_1 &= -\frac{\lambda}{4!}\Phi^4.
\end{aligned} \tag{1.27}$$

We will work both in the Minkowski and Euclidian space upon convenience.

The generating functional in the euclidian space has the expression:

$$W[J] = \int d\Phi \exp[-\int d^4x[\frac{1}{2}(\frac{\partial\Phi}{\partial\tau})^2 + \frac{1}{2}(\Delta\Phi)^2 + \frac{1}{2}m_0^2\Phi^2 + \frac{\lambda}{4!}\Phi^4 + J\Phi]] \tag{1.28}$$

and can be written as:

$$W[J] = \exp[\int d^4x \mathcal{L}_1(\frac{\delta}{\delta J})] W_0[J] \tag{1.29}$$

where,

$$W_0[J] = \int d\Phi \exp[\int d^4x(\mathcal{L}_0 + J\Phi)]. \tag{1.30}$$

From Eq.(1.29) is clear how the perturbative approach can work. If λ is a small parameter one can expand the exponential in terms of λ and solve succesive contributions accordingly. However we are interested in the regime where λ is large and one cannot use the above expansion.

We will illustrate our approach simply on a simple function. Assume we have the following one-dimensional integral which cannot be solved analytically:

$$I = \int dx \exp[-af(x)], \tag{1.31}$$

where f is polynomial of x. For a small the expansion in a makes sense. For $a \to \infty$ the Taylor expansion uses:

$$\lim_{a\to\infty} \frac{d^n \exp[-af(x)]}{da^n} = 0 \tag{1.32}$$

which does not lead to a correct answer.

We shall use however a simple trick. We replace in the polynomial f some of the variables x with a new variable y (for example $x^4 \to x^2y^2$). Then we write:

$$\begin{aligned} I &= \int dxdy\delta(x-y)exp[-af(x,y)] = \int dxdydz \exp[-i(x-y)z] \exp[-af(x,y)] = \\ &\int dxdydz \exp[-i(x-y)z - af(x,y)] \end{aligned} \tag{1.33}$$

This does not help too much in the present form. However if $f(x,y) = x^2y^2$ or any other function that contains x^2 we can form the perfect square:

$$-ixz - ax^2y^2 = -(\sqrt{a}xy + \frac{iz}{2\sqrt{a}y})^2 - \frac{z^2}{4ay^2}. \tag{1.34}$$

Introduced in Eq. (1.122) this leads:

$$I = \text{const} \int d\frac{1}{\sqrt{a}y} dz \exp[-\frac{z^2}{4ay^2}] \exp[iyz] \tag{1.35}$$

Then expansion in $\frac{1}{a}$ makes sense and one can write:

$$I = \text{const} \int dxdz \frac{1}{\sqrt{ay}}[1 - \frac{z^2}{4ay^2} + ...] \exp[iyz] \tag{1.36}$$

This expansion may seem ill defined and highly divergent. For example if one integrates over z already encounters infinities. However in the functional method one is dealing with functions instead of simple variables and one encounters divergences also in the usual expansion in small parameters. Such that we will consider the above approach as our starting point and solve the problem of divergences as they appear.

We will start with the simple partition function for a Φ^4 theory without a source:

$$W[0] = \int d\Phi \exp[i \int d^4x[\mathcal{L}_0 + \mathcal{L}_1]] \tag{1.37}$$

We consider the extended functional δ defined in the Minkowski space as (see the Appendix):

$$\delta(\Phi) = \text{const} \int dK \exp[i \int d^4x_M K\Phi] \tag{1.38}$$

which in the euclidian space becomes:

$$\delta(\Phi) = \text{const} \int dK \exp[- \int d^4x K\Phi] \tag{1.39}$$

We then rewrite Eq. (1.37)in Minkowski space as:

$$\begin{aligned}
&W[0] = \int d\Phi d\Psi \delta(\Phi - \Psi) \exp[i \int d^4x[\mathcal{L}_0 - \frac{\lambda}{8}\Phi^2\Psi^2]] = \\
&\text{const} \int d\Phi d\Psi dK \exp[i \int d^4x K(\Phi - \Psi)] \exp[i \int d^4x[\mathcal{L}_0 - \frac{\lambda}{8}\Phi^2\Psi^2]] = \\
&\text{const} \int \frac{1}{\sqrt{\lambda}} d\Phi dK \exp[i \int d^4x \frac{2}{\lambda}K^2] \exp[i \int d^4x K\Phi^2] \exp[i \int d^4x \mathcal{L}_0]
\end{aligned} \tag{1.40}$$

In order to obtain this result we made the following change of variable in the second line of Eq. (1.40): $K \to K\Phi$, $\Psi \to \frac{\Psi}{\Phi\sqrt{\lambda}}$. Note that the λ term gets rescaled by 3 such that to take into account the various contribution of the Fourier modes.

We will estimate the first order of the integral in Eq. (1.40) given by:

$$\text{const} \int \frac{1}{\sqrt{\lambda}} d\Phi dK \frac{1}{\exp}[i \int d^4x K\Phi^2] \exp[i \int d^4x \mathcal{L}_0] \tag{1.41}$$

In order to solve the integral we write:

$$\int d^4x[K\Phi^2+\mathcal{L}_0]=\int d^4x[\frac{1}{V^2}\sum_{k_n+k_m+k_p=0}(\mathrm{Re}K_m+i\mathrm{Im}K_m)(\mathrm{Re}\Phi_n+i\mathrm{Im}\Phi_n)(\mathrm{Re}\Phi_p+i\mathrm{Im}\Phi_p)-$$
$$-\frac{1}{2V}\sum_{k_n}(m_0^2-k_n^2)[(\mathrm{Re}\Phi_n)^2+(\mathrm{Im}\Phi_n)^2]] \tag{1.42}$$

We denote the bilinear form in the exponential in the Eq.(1.42) by:

$$\Phi[\frac{K}{V^2}-\frac{1}{2V}[\frac{2K_0}{V}-(m_0^2-p_n^2)(\delta_{2n+1,2n+1}+\delta_{2n+2,2n+2})]]\Phi \tag{1.43}$$

where the counting starts from $n=0$ and we arranged for example the $\mathrm{Re}\Phi_n$ and $\mathrm{Im}\Phi_n$ components in the $2n+1$, respectively $2n+2$ columns of an infinitely dimensional vector.

Then the integral in Eq. (1.41) can be solved easily as a gaussian integral:

$$\begin{aligned}&\mathrm{const}\int\frac{1}{\sqrt{\lambda}}d\Phi dK\exp[i\int d^4xK\Phi^2]\exp[i\int d^4x\mathcal{L}_0]=\\&=\int dK\frac{1}{\det[\frac{K}{V^2}+\frac{1}{2V}[\frac{2K_0}{V}-(m_0^2-p_n^2)(\delta_{2n+1,2n+1}+\delta_{2n+2,2n+2})]]^{1/2}}.\end{aligned} \tag{1.44}$$

Note that one can write also a result for the full partition function in Eq. (1.40):

$$\begin{aligned}&\mathrm{const}\int\frac{1}{\sqrt{\lambda}}d\Phi dK\exp[i\int d^4x\frac{2}{\lambda}K^2]\exp[i\int d^4xK\Phi^2]\exp[i\int d^4x\mathcal{L}_0]=\\&=\int dK\exp[i\int d^4x\frac{2}{\lambda}K^2]\frac{1}{\det[\frac{K}{V^2}+\frac{1}{2V}[\frac{2K_0}{V}-(m_0^2-p_n^2)(\delta_{2n+1,2n+1}+\delta_{2n+2,2n+2})]]^{1/2}}.\end{aligned} \tag{1.45}$$

The next step is to determine through this procedure the propagator.This is is given by:

$$\langle\Omega|T\Phi(x_1)\Phi(x_2)|\Omega\rangle=\lim_{T\to\infty}\frac{\int d\Phi\Phi(x_1)\Phi(x_2)\exp[i\int_{-T}^{T}d^4x\mathcal{L}]}{\int d\Phi\exp[i\int_{-T}^{T}d^4x\mathcal{L}]}. \tag{1.46}$$

For our partition function the Eq. (1.46) is rewritten as:

$$\begin{aligned}&\langle\Omega|T\Phi(x_1)\Phi(x_2)|\Omega\rangle=\\&\frac{\int\frac{1}{\sqrt{\lambda}}d\Phi dK\Phi(x_1)\Phi(x_2)\exp[i\int d^4x\frac{2}{\lambda}K^2]\exp[i\int d^4xK\Phi^2]\exp[i\int d^4x\mathcal{L}_0]}{\int\frac{1}{\sqrt{\lambda}}d\Phi dK\exp[i\int d^4x\frac{2}{\lambda}K^2]\exp[i\int d^4xK\Phi^2]\exp[i\int d^4x\mathcal{L}_0]}=\\&\frac{1}{V^2}\sum_m\exp[-ip_m(x_1-x_2)]iV\times\\&\frac{\frac{\delta}{\delta(m^2-p_m^2)}\int d\Phi dK\exp[i\int d^4x\frac{2}{\lambda}K^2]\exp[i\int d^4xK\Phi^2]\exp[id^4x\mathcal{L}_0]}{\int d\Phi dK\exp[i\int d^4x\frac{2}{\lambda}K^2]\exp[i\int d^4xK\Phi^2]\exp[id^4x\mathcal{L}_0]}\end{aligned} \tag{1.47}$$

Note that the first line in Eq. (1.47) is the standard definition of the two point function. The second line in Eq. (1.47) needs some clarification. From the first line in the equation it can be seen that the scalar two point function may receive contributions either from the kinetic term or from the terms that contain K. We need to show that also the second line is justified. In order to see that one should consider the simple functional integral in Eq. (1.47) and treat it independently without any reference to the Feynman diagrams. Then the first line of Eq. (1.47) leads also to:

$$
\begin{aligned}
&\langle\Omega|T\Phi(x_1)\Phi(x_2)|\Omega\rangle = \frac{\int \frac{1}{\sqrt{\lambda}} d\Phi dK \Phi(x_1)\Phi(x_2)\exp[i\int d^4x \frac{2}{\lambda}K^2]\exp[i\int d^4x K\Phi^2]\exp[i\int d^4x\mathcal{L}_0]}{\int \frac{1}{\sqrt{\lambda}} d\Phi dK \exp[i\int d^4x \frac{2}{\lambda}K^2]\exp[i\int d^4x K\Phi^2]\exp[i\int d^4x\mathcal{L}_0]} = \\
&\frac{1}{V^2}\sum_m\sum_n \exp[-ip_m x_1]\exp[-ip_n x_2] iV^2 \times \\
&\frac{\int d\Phi dK \exp[i\int d^4x\frac{2}{\lambda}K^2]\frac{\delta}{\delta K(-p_m-p_n)}\exp[i\int d^4x K\Phi^2]\exp[id^4x\mathcal{L}_0]}{\int d\Phi dK \exp[i\int d^4x\frac{2}{\lambda}K^2]\exp[i\int d^4x K\Phi^2]\exp[id^4x\mathcal{L}_0]} = \\
&\frac{1}{V^2}\sum_m\sum_n \exp[-ip_m x_1]\exp[-ip_n x_2](-i)V^2 \times \\
&\frac{\int d\Phi dK \frac{\delta}{\delta K(-p_m-p_n)}\exp[i\int d^4x\frac{2}{\lambda}K^2]\exp[i\int d^4x K\Phi^2]\exp[id^4x\mathcal{L}_0]}{\int d\Phi dK \exp[i\int d^4x\frac{2}{\lambda}K^2]\exp[i\int d^4x K\Phi^2]\exp[id^4x\mathcal{L}_0]} = \\
&\frac{1}{V^2}\sum_m\sum_n \exp[-ip_m x_1]\exp[-ip_n x_2](-\frac{2}{\lambda})V \times \\
&\frac{\int d\Phi dK K(p_n+p_m)\exp[i\int d^4x\frac{2}{\lambda}K^2]\exp[i\int d^4x K\Phi^2]\exp[id^4x\mathcal{L}_0]}{\int d\Phi dK \exp[i\int d^4x\frac{2}{\lambda}K^2]\exp[i\int d^4x K\Phi^2]\exp[id^4x\mathcal{L}_0]}
\end{aligned}
\tag{1.48}
$$

We shall attempt to estimate the integral over the modes K(p) in the Eq. (1.175). For that we expand the exponential of the trilinear term. In first order we get the term,

$$
K(p_m+p_n)K(-q-r)\Phi(q)\Phi(r) \tag{1.49}
$$

which is evident that brings contribution only for $q=-r$, $p_m=-p_n$ so the only K mode that contributes is the zero mode. In third order order (second order is zero) we obtain terms of the type:

$$
K(p_m+p_n)K(-q_1-r_1)K(-q_2-r_2)K(-q_3-r_3)\Phi(q_1)\Phi(r_1)\Phi(q_2)\Phi(r_2)\Phi(q_3)\Phi(r_3) \tag{1.50}
$$

If any of the $q_i=-r_i$ we are back to the previous case where only K_0 mode contribute. Assume without loss of generality that $q_1=-q_2$, $r_1=-r_3$, $q_3=-r_2$. This settle the integral over Φ whereas for K we obtain:

$$
K(p_m+p_n)K(-q_1-r_1)K(q_1-r_2)K(r_1+r_2) \tag{1.51}
$$

There are three possibilities for this integral: 1) $p_m + p_n = q_1 + r_1$, $q_1 - r_2 = -r_1 - r_2$, 2) $p_m + p_n = -q_1 + r_2$, $q_1 + r_1 = r_1 + r_2$, 3) $p_m + p_n = -r_1 - r_2$, $q_1 + r_1 = q_1 - r_2$. All of these possibilities lead to $p_n = -p_m$. This arguments continues for higher orders in the expansion such that quite justified we can express the propagator from the beginning as the derivative with respect to $m^2 - p_m^2$.

Since the quantity $m^2 - k_m^2$ appears only in the determinant in Eq. (1.45) we can compute:

$$
\begin{aligned}
&\frac{\delta}{\delta(m_0^2 - p_m^2)}[\det[\frac{K}{V^2} + \frac{1}{2V}[\frac{2K_0}{V} - (m_0^2 - p_n^2)(\delta_{2n+1,2n+1} + \delta_{2n+2,2n+2})]]]^{-1/2} = \\
&-\frac{1}{2}[\det[\frac{K}{V^2} - \frac{1}{2V}[\frac{2K_0}{V} - (m_0^2 - p_n^2)(\delta_{2n+1,2n+1} + \delta_{2n+2,2n+2})]]]^{-1/2} \times \\
&Tr[\frac{1}{\frac{K}{V^2} + \frac{1}{2V}[2\frac{K_0}{V} - (m_0^2 - p_n^2)(\delta_{2n+1,2n+1} + \delta_{2n+2,2n+2})]}(-1)(\frac{1}{2V}(\delta_{2m+1,2m+1} + \delta_{2m+2,2m+2}))] = \\
&\text{const}\frac{1}{2}[\det[\frac{K}{V^2} - \frac{1}{2V}[2\frac{K_0}{V} - (m_0^2 - p_n^2)(\delta_{2n+1,2n+1} + \delta_{2n+2,2n+2})]]]^{-1/2}\frac{2}{\frac{2}{V}K_0 - (m_0^2 - p_m^2)} = \\
&\text{const}[\det[\frac{K}{V^2} - \frac{1}{2V}[2K_0 - (m_0^2 - p_n^2)(\delta_{2n+1,2n+1} + \delta_{2n+2,2n+2})]]]^{-1/2}\frac{1}{\frac{2K_0}{V} - (m_0^2 - p_m^2)}.
\end{aligned}
\tag{1.52}
$$

In Eq. (1.52) the first three lines are the simple result of differentiating a determinant. The first factor in the third line of Eq. (1.52) contains the Fourier modes of the field K with momenta different than zero ($p_\mu \neq 0$) denoted simply by K and those with momenta $p_\mu = 0$ denoted by K_0. However the modes with $p_\mu \neq 0$ are irrelevant for the reason we shall outline below. First let us consider $K(x)$ as a square integrable function in the Hilbert space which satisfies:

$$\int d^4x K^2(x) = \frac{1}{V}\sum_p K(p)^2 < M, \tag{1.53}$$

where M is a quantity large but finite. This means that $\frac{K(p)}{V} < \frac{\sqrt{M}}{\sqrt{V}}$. In contrast $\frac{K_0}{V}$ is finite as is given by:

$$\frac{K_0}{V} = \int d^4x K(x) \tag{1.54}$$

We could have dropped from the beginning the factor $\frac{1}{V}$ from its expression but it helps with dimensional analysis. Thus although we shall keep $K(p \neq 0)$ in the expression at some point the limit $V \to \infty$ will be taken such that all these terms in the determinant will cancel and the integral of the exponential of the $K(p_n)$ terms in the numerator will get canceled by that in the denominator. In conclusion the zero mode is used as a substitute for all the the other modes and sums up all their contribution.

The mode K_0 acts like an additional contribution to the scalar mass and needs to be maintained and integrated over. In consequence in all calculations that follows one should consider only the modes K_0 facts which simplifies the calculations considerably.

Then Eq. (1.47) becomes:

$$\langle\Omega|T\Phi(x_1)\Phi(x_2)|\Omega\rangle = \frac{1}{V^2}\sum_m \exp[-ip_m(x_1-x_2)]iV \times$$
$$\times\frac{\int dK \frac{1}{\frac{2}{V}K_0-(m_0^2-p_m^2)}\exp[i\int d^4x\frac{2}{\lambda}K^2]\frac{1}{\det[\frac{K}{V^2}+\frac{1}{2V}[2\frac{K_0}{V}-(m_0^2-p_n^2)(\delta_{2n+1,2n+1}+\delta_{2n+2,2n+2})]]^{1/2}}}{\int dK\exp[i\int d^4x\frac{2}{\lambda}K^2]\frac{1}{\det[\frac{K}{V^2}+\frac{1}{2V}[\frac{2K_0}{V}-(m_0^2-p_n^2)(\delta_{2n+1,2n+1}+\delta_{2n+2,2n+2})]]^{1/2}}} \quad (1.55)$$

We denote:

$$\frac{1}{\lambda}\frac{2}{V} = ba_0$$
$$m_0^2 - p_m^2 = c^2$$
$$\frac{2}{V} = a_0$$
$$\det[\frac{K}{V^2}+\frac{1}{2V}[\frac{2K_0}{V}-(m_0^2-p_n^2)(\delta_{2n+1,2n+1}+\delta_{2n+2,2n+2})]] = \det[a_0K_0+B] \quad (1.56)$$

We need to evaluate:

$$\frac{\int dKdK_0\exp[2iba_0K_0^2+\int d^4x2ibK^2]\frac{1}{(a_0K_0-c^2)[\det[a_0K_0+B]]^{1/2}}}{\int dKdK_0\exp[ibK_0^2+\int d^4xibK^2]\frac{1}{[\det[a_0K_0+B]]^{1/2}}} =$$
$$-\frac{1}{c^2}\frac{\int dKdK_0\exp[2iba_0K_0^2+\int d^4x2ibK^2][1+\frac{a_0K_0}{c^2}+\frac{a_0^2K_0^2}{c^4}+...]\frac{1}{[\det[a_0K_0+B]]^{1/2}}}{\int dKdK_0\exp[ibK_0^2+\int d^4xibK^2]\frac{1}{[\det[a_0K_0+B]]^{1/2}}} \quad (1.57)$$

We extracted a factor of $\frac{1}{V}$ from the determinant and dropped the corresponding constant factor everywhere. In order to determine the ratio in Eq. (1.67) we evaluate each term in the expansion in the denominator:

$$I_n = \int dK_0dK\frac{(a_0K_0)^n}{c^{2n}}\exp[2iba_0K_0^2]\exp[\int d^4x2ibK^2](\det[a_0K_0+B])^{-1/2} =$$
$$\int dK_0dK\frac{1}{a_0}\frac{d[\frac{(a_0K_0)^{n+1}}{c^{2n}(n+1)}]}{dK_0}\exp[2iba_0K_0^2]\exp[\int d^4x2ibK^2](\det[aK_0+B])^{-1/2} =$$
$$-\int dK_0dK\frac{1}{a_0}\frac{(a_0K_0)^{n+1}}{c^{2n}(n+1)}(4iba_0K_0)\exp[2iba_0K_0^2]\exp[\int d^4x2ibK^2](\det[aK_0+B])^{-1/2} -$$
$$\int dK_0dK\frac{(a_0K_0)^{n+1}}{a_0c^{2n}(n+1)}\exp[2iba_0K_0^2]\sum_k[\frac{-a_0}{a_0K_0-c_k^2}]\exp[\int d^4x2ibK^2](\det[a_0K_0+B])^{-1/2} =$$
$$-\frac{4ibc^4}{a_0(n+1)}I_{n+2} - \int dKdK_0\frac{(a_0K_0)^{n+1}}{c^{2n}(n+1)}\sum_k\frac{1}{c_k^2}[1+\frac{a_0K_0}{c_k^2}+\frac{(a_0K_0)^2}{c_k^4}+...]\times$$
$$\exp[2iba_0K_0^2]\exp[\int d^4x2ibK^2](\det[a_0K_0+B])^{-1/2}. \quad (1.58)$$

Here we used the formula of differentiation of a determinant.

From Eqs. (1.175) and (1.65) we obtain the following recurrence formula:

$$(n+1)I_n + I_{n+2}c^4[\frac{4ib}{a_0} + \sum_k \frac{1}{c_k^4}] + I_{n+1}c^2 \sum_k \frac{1}{c_k^2} + ... + I_{n+r}c^{2r} \sum_k \frac{1}{c_k^{2r}} + ... = 0 \quad (1.59)$$

First we multiply the whole Eq. (1.59) by $\frac{1}{V}$ and then introduce $I_n c^{2n} = J_n$ to get the new recurrence formula:

$$\frac{1}{V}(n+1)J_n + J_{n+1}\frac{1}{V}\sum_k \frac{1}{c_k^2} + J_{n+2}[2ib + \frac{1}{V}\sum_k \frac{1}{c_k^4}] + ... = 0 \quad (1.60)$$

Finally since we denoted the partition function by I_0 from Eqs. (1.74) and (1.60) one can derive:

$$\begin{aligned} \text{Propagator} &= -\frac{1}{c^2}\sum_n I_n/I_0 = \\ &= -\frac{1}{c^2}\sum_n \frac{1}{c^{2n}} J_n/I_0, \end{aligned} \quad (1.61)$$

where $J_0 = I_0$ is the full partition function.

Before going further we need to determine the coefficients in Eq. (1.60). For that we first state,

$$\frac{1}{V}\sum_k \frac{1}{c_k^{2r}} = \frac{1}{V}\sum_k \frac{1}{(m_0^2 - p_k^2)^r} = (-1)^r \int d^4p \frac{1}{(p^2 - m_0^2)^r} = q_r. \quad (1.62)$$

Note that only the integral with $k = 1, 2$ are divergent whereas the other ones are finite. We shall use a simple cut-off the regularize them upon the case. Then we get:

$$\begin{aligned} q_1 &= i\frac{1}{16\pi^2}[\Lambda^2 - m_0^2 \ln[\frac{\Lambda^2}{m_0^2}]] \\ q_2 &= i\frac{1}{16\pi^2}[-1 + \ln[\frac{\Lambda^2}{m_0^2}]] \\ q_{n,n>2} &= i\frac{1}{16\pi^2}\frac{(m_0^2)^{2-n}}{(n-1)(n-2)}. \end{aligned} \quad (1.63)$$

The terms J_n in the two point function in Eq. (1.61) correspond to various loop corrections and one can cut the series to obtain results in various orders of perturbation theory. However we shall not attempt to do this here. We will rather aim to obtain if possible an all order result for the correction to the mass of the scalar. We do this with the hope that the approach initated here can be extended easily to theories with spontaneous symmetry breaking and even to the standard model. It is clear that an approach that could estimate the correction to the Higgs boson mass could prove of great interest. One can write quite generally an exact expression for the propagator of a scalar:

$$\frac{i}{p^2 - m^2 - M^2(p^2)} \tag{1.64}$$

where m is the physical mass and $M^2(p^2)$ is the one particle irreducible self energy (For simplicity we rename $p_m^2 = p^2$ for the rest of the paper). In our approach the propagator is given by:

$$\frac{i}{p^2 - m_0^2}\sum_n (-1)^n \frac{J_n}{I_0}\frac{1}{(p^2 - m_0^2)^n} \tag{1.65}$$

Now if we identify Eq. (1.64) with Eq. (1.65) and expand the first equation in series in $\frac{1}{(p^2-m_0^2)^n}$ we obtain :

$$\begin{aligned}
&\frac{i}{p^2 - m_0^2}\sum_n (-1)^n \frac{J_n}{I_0}\frac{1}{(p^2 - m_0^2)^n} = \frac{i}{p^2 - m^2 - M^2(p^2)}\\
&(-1)^n \frac{J_n}{I_0} = [m^2 - m_0^2 - M^2(p^2)]^n + Y_n(p^2)\\
&\frac{J_n}{I_0} = [m_0^2 - m^2 - M^2(p^2)]^n + (-1)^n Y_n(p^2),
\end{aligned} \tag{1.66}$$

where $Y_n(p^2)$ is an arbitrary series with the property:

$$\sum_n Y_n(p^2)\frac{1}{(p^2 - m_0^2)^{n+1}} = 0 \tag{1.67}$$

Now we shall consider the following renormalization conditions which states:

$$\begin{aligned}
&M^2(p^2)_{p^2=m^2} = 0\\
&\frac{dM^2(p^2)}{dp^2}|_{p^2=m^2} = 0
\end{aligned} \tag{1.68}$$

We apply the first condition to Eq. (1.66) to determine that:

$$\frac{(-1)^n J_n}{I_0}|_{p^2=m^2} = (m^2 - m_0^2)^n + Y_n(m^2) = (m^2 - m_0^2)^n (1 + \frac{Y_n}{(m^2 - m_0^2)^n}) \tag{1.69}$$

We will show that the term $Y_n(m^2)$ in Eq. (1.69) should be set to zero. For that we first note that from Eq. (1.67) one can deduce that there is at least one n for which $\frac{Y_n(m^2)}{(m^2-m_0^2)^n} < 0$. Then there is a solution m for which $(m^2 - m_0^2)^n = -\frac{1}{Y_n(m^2)} = \alpha(n)^n$ with $\alpha(n)$ real. This solution is a zero of the corresponding J_n. But $J_n(m^2)$ has the expression:

$$\int dK_0 K_0^n \frac{1}{a_0 K_0 - \alpha(n)} \times \text{other factors}, \tag{1.70}$$

so has a pole at $a_0K_0 = \alpha(n)$ instead of a zero. We obtain a contradiction which means that there is no n such that $\frac{Y_n(m^2)}{(m^2-m_0^2)^n} < 0$ so the series in Eq. (1.67) has all the terms $Y_n(m^2) = 0$.

Then we simply take:

$$\frac{J_n}{I_0}|_{p^2=m^2} = [m_0^2 - m^2]^n. \tag{1.71}$$

We denote,

$$X = [m_0^2 - m^2], \tag{1.72}$$

and sum in the recurrence formula in Eq. (1.60) all terms with the indices $n+k$, $k \geq 3$ for $p^2 = m^2$.

$$\begin{aligned}&\sum_{k\geq 3}\frac{J_{n+k}}{I_0}q_k = X^n\sum_k \frac{i}{16\pi^2}m_0^4(\frac{X}{m_0^2})^n\frac{1}{(n-1)(n-2)} = \\ &\frac{i}{16\pi^2}X^{n+1}[X + (m_0^2 - X)\ln[\frac{m_0^2 - X}{m_0^2}]]\end{aligned} \tag{1.73}$$

Then the recurrence formula becomes:

$$\begin{aligned}&(n+1)a_0X^n + q_1X^{n+1} + (\frac{2i}{\lambda} + q_2)X^{n+2} + \frac{i}{16\pi^2}X^{n+1}[X + (m_0^2 - X)\ln[\frac{m_0^2 - X}{m_0^2}]] = 0\\ &(n+1)a_0\frac{1}{X} + q_1 + (\frac{2i}{\lambda} + q_2)X + \frac{i}{16\pi^2}[X + (m_0^2 - X)\ln[\frac{m_0^2 - X}{m_0^2}]] = 0\\ &q_1 + (\frac{2i}{\lambda} + q_2)X + \frac{i}{16\pi^2}[X + (m_0^2 - X)\ln[\frac{m_0^2 - X}{m_0^2}]] = 0.\end{aligned} \tag{1.74}$$

Here in the last line we took the limit $a_0 = \frac{1}{V} \to 0$.

Note that although we used the conditions in Eq. (1.68) we should not consider our approach equivalent with any of the standard renormalization procedures.

Then Eq. (1.74) will become:

$$q_1 + (m_0^2 - m^2)[\frac{2i}{\lambda} + q_2] + \frac{i}{16\pi^2}[(m_0^2 - m^2) + m^2\ln[\frac{m^2}{m_0^2}]] = 0 \tag{1.75}$$

The equation above determines the physical mass in terms of the bare mass and of the cut-off scale. Instead we observe that for a large cut-off scale one can divide the Eq. (1.75) by q_1 and retain the first and second term. Then,

$$m^2 \approx m_0^2 + \frac{q_1}{\frac{2i}{\lambda} + q_2} \approx m_0^2 + \frac{\Lambda^2 - m_0^2\ln[\frac{\Lambda^2}{m_0^2}]}{1 + \frac{\lambda}{32\pi^2}[-1 + \ln[\frac{\Lambda^2}{m_0^2}]]}\frac{\lambda}{32\pi^2}. \tag{1.76}$$

Note that this result leads to the same first order coefficient of the mass anomalous dimension as in the standard renormalization procedures.

We shall present next an approximate estimate of the propagator for an arbitrary regularization scheme for the limit of large coupling λ. We start from the recurrence relation in Eq. (1.60) which we rewrite here for completeness:

$$\frac{1}{V}(n+1)J_n + J_{n+1}\frac{1}{V}\sum_k \frac{1}{c_k^2} + J_{n+2}[2ib + \frac{1}{V}\sum_k \frac{1}{c_k^4}] + ... = 0 \tag{1.77}$$

We denote:

$$\frac{1}{V}\sum_k \frac{1}{(c_k^2)^n} = s_n, \tag{1.78}$$

where s_n may be considered in any regularization scheme. First we will make a change of variable $K_0 = \frac{K_0'}{\sqrt{\lambda}}$ and rewrite $J_n = \frac{S_n}{\lambda^{n/2}}$ where S_n represent the same quantity as J_n this time in the variable K_0'. Note that with this change of notation for λ very large the factor $\exp[2iba_0\frac{K_0'^2}{\lambda}]$ in Eq. (1.175) becomes negligible.

The recurrence formula becomes in terms of S_n:

$$\begin{aligned}
&\frac{1}{V}(n+1)S_n/\lambda^{n/2} + S_{n+1}/\lambda^{(n+1)/2}s_1 + \\
&S_{n+2}/\lambda^{(n+2)/2}[2ib + s_2] + S_{n+3}/\lambda^{(n+3)/2}s_3 ... = 0
\end{aligned} \tag{1.79}$$

For large V and λ one obtains in first order:

$$S_{n+2} = -S_{n+1}\frac{\sqrt{\lambda}s_1}{2ib + s_2} \tag{1.80}$$

to determine $S_n = (-1)^{n-1}S_1[\frac{\sqrt{\lambda}s_1}{2ib+s_2}]^{n-1} = (-1)^{n-1}x^{n-1}\lambda^{(n-1)/2}S_1$.

In order to determine S_1 we consider the zeroth order recurrence relation:

$$\frac{1}{V}S_0 + \sum_{n=3} S_1(-1)^{n-1}x^{n-1}\lambda^{(n-1)/2}/\lambda^{n/2}s_n = 0 \tag{1.81}$$

This yields:

$$S_1 = -\frac{1}{V}\sqrt{\lambda}S_0/[\sum_{n=3}(-1)^{n-1}x^{n-1}s_n] \tag{1.82}$$

According to Eq. (1.61) the propagator is given by:

$$\begin{aligned}
&\text{Propagator} = -\frac{1}{c^2}\sum_n \frac{1}{c^{2n}}(S_n\lambda^{-n/2})/S_0 = \\
&= -\frac{1}{c^2}[S_0 + \sum_{n=1}\frac{1}{c^{2n}}(-1)^{n-1}x^{n-1}\lambda^{(n-1)/2}\lambda^{-n/2}S_1]/S_0 = -\frac{1}{c^2}[S_0 + \frac{1}{\sqrt{\lambda}(c^2+x)}S_1]/S_0 = \\
&-\frac{1}{c^2}[1 - \frac{1}{c^2+x}\frac{1}{V[\sum_{n=3}(-1)^{n-1}x^{n-1}s_n]}],
\end{aligned} \tag{1.83}$$

where $x = \frac{s_1}{2ib+s_2}$ (See the notation in Eq. (1.56)). A straightforward computation for the quantities in Eq. (1.83) in the limit of large λ leads to:

$$\text{Propagator} \approx -\frac{1}{c^2}[1 - \Lambda_1^2 \frac{1}{c^2 + \Lambda^2}], \tag{1.84}$$

where $\Lambda_1^2 = \text{function of}(\Lambda^2) < \Lambda^2$. Note that in the notation in the paper $c^2 = m_0^2 - p^2$.

This is a particular case of " triviality" known to be a feature of the Φ^4 theories in the limit where $\lambda \to \infty$. To show this we rewrite the Eq. (1.84) as:

$$\begin{aligned}\text{Propagator} &= \frac{1}{p^2 - m_0^2}[1 - \Lambda_1^2 \frac{1}{m_0^2 + \Lambda^2 - p^2}] = \\ &= \frac{\Lambda^2 - \Lambda_1^2}{\Lambda^2} \frac{1}{p^2 - m_0^2} + \frac{\Lambda_1^2}{\Lambda^2} \frac{1}{p^2 - m_0^2 - \Lambda^2}\end{aligned} \tag{1.85}$$

According to some authors a theory is trivial in the strong coupling regime if the propagator can be written as:

$$\text{Propagator} = \sum_n \frac{Z_n}{p^2 - m_n^2} \tag{1.86}$$

where Z_n (all of them can be zero except one) are the weights and m_n are the spectrum in the large coupling limit. As it can be observed easily our result in Eq. (1.85) is a particular case of triviality with the masses $m_1^2 = m_0^2$ and $m_2^2 = m_0^2 + \Lambda^2$.

C. Partition function for a gauge theory with fermions in a new functional approach.

We will used the method introduced in the preceding sections as a laboratory for studying more complex theories; in the present work we shall apply the method illustrated previously to the case of QED with fermions in the fundamental representation. We will find out that the beta function in this approach has only the first two orders coefficients different than zero and thus corresponds to a renormalization scheme long time suggested by 't Hooft . This renormalization scheme was investigated by Garkusha and Kataev for a nonabelian gauge theory where they found that the 't Hooft scheme might hinder some theoretical effects enclosed in the higher order corrections to the Green functions and to the beta function. In doing this the authors consider the connection that exists between the $\overline{MS}$ scheme and the 't Hooft scheme where the corresponding beta function is obtained by finite charge renormalization. Here we need to make some amends regarding our procedure for determining the beta function. The method we use is a method of pure functional integration that has only up to a point a counterpart in the Feynman diagram formalism. In order to determine the beta function some estimate of the integral functional are made and some assumptions are used. Thus although the result is that expected the method is in many instances new. However this does not contradicts the findings that in this procedure some theoretical aspects are lost.

The QED Lagrangian displays more complicated interactions and set-up. We start with the Lagrangian,

$$\mathcal{L}_{QED} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.87}$$

We rewrite the Lagrangian in Eq. (1.87) in terms of the Fourier modes because we would like to integrate over these in the functional approach:

$$\begin{aligned}&\int d^4x\mathcal{L}_{QED} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}A_\mu(k)[-k^2g^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu]A_\nu(-k) + \\ &\int \frac{d^4k}{(2\pi)^4}[\Psi(k)(\gamma^\mu k_\mu - m)\Psi(-k)] - e\int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\bar{\Psi}(p)\gamma^\mu\Psi(-p+k)A_\nu(-k),\end{aligned} \tag{1.88}$$

where we count over both positive and negative k modes.

We shall integrate expression in (1.88) following the method introduced before. We consider the function of the Fourier modes of the gauge field as a quadratic form plus a linear term. By forming the corresponding gaussian form we integrate over the gauge fields to obtain:

$$\begin{aligned}&W[0] = \int d\Psi d\bar{\Psi} dA_\mu \exp[i\int d^4x\mathcal{L}_{QED}] = \\ &\text{const}\int\prod_i\prod_j d\Psi(p_i)d\bar{\Psi}(p_j)(\det[k^2g^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu])^{-1/2}\times \\ &\exp[i[\int\frac{d^4k}{(2\pi)^4}[\bar{\Psi}(k)(\gamma^\mu k_\mu - m)\Psi(-k)] - \\ &\int e^2\frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}[\frac{1}{4}\bar{\Psi}(p)\gamma^\mu\Psi(p+k)D^{-1}_{\mu\nu}\bar{\Psi}(q)\gamma^\nu\Psi(q-k)]].\end{aligned} \tag{1.89}$$

Here we made the change of variables $A_\mu(k) \to A_\mu(k) - \frac{\bar{\Psi}(p)\gamma^\nu\Psi(-p+k)}{2D^{\mu\nu}}$ and denoted :

$$D^{\mu\nu} = -k^2g^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu, \tag{1.90}$$

and ξ is the usual gauge parameter. Moreover we counted only over the modes with $k_0 > 0$ case in which the kinetic term for the gauge field appears without the factor of $\frac{1}{2}$ in front. We shall integrate in Eq. (1.89) by introducing a new variable η_μ and a delta function:

$$W[0] = \text{const}\int\prod_i\prod_j\prod_k d\bar{\Psi}(p_i)d\Psi(p_j)d\eta_\mu(p_k)(\det[k^2g^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu])^{-1/2}\delta(\eta_\mu - \bar{\Psi}\gamma^\mu\Psi)\times$$
$$\exp[i[\int\frac{d^4k}{(2\pi)^4}\bar{\Psi}(k)(\gamma^\mu k_\mu - m)\Psi(-k) - \int\frac{e^2}{4}\frac{d^4k}{(2\pi)^4}\eta^\mu D^{-1}_{\mu\nu}\eta^\nu]] \tag{1.91}$$

We further express the delta function in terms of its exponential representation to get:

$$W[0] = \text{const}\int\prod_i\prod_j\prod_k\prod_r d\bar{\Psi}(p_i)d\Psi(p_j)d\eta_\mu(p_k)dK_\mu(p_r)(\det[-i(k^2g^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu)])^{-1/2}\times$$
$$\exp[i\int\frac{d^4k}{(2\pi)^4}K_\mu(p)(\eta_\mu(-p) - \bar{\Psi}(q)\gamma^\mu\Psi(-q-p))]\times$$
$$\exp[i[\int\frac{d^4k}{(2\pi)^4}\bar{\Psi}(k)(\gamma^\mu k_\mu - m)\Psi(-k) - \int\frac{e^2}{4}\frac{d^4k}{(2\pi)^4}\eta^\mu(k) D^{-1}_{\mu\nu}\eta^\nu(-k)]] \tag{1.92}$$

We then integrate over the η_μ field by forming quadratic forms out of the expression in the exponent. This leads to:

$$W[0] = \text{const}\int\prod_i\prod_j\prod_k d\bar{\Psi}(p_i)d\Psi(p_j)dK_\mu(p_k)(\det[-i(k^2g^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu)])^{-1/2}\times$$
$$(\det[\frac{1}{e^2}D_{\mu\nu}])^{1/2}\exp[i\int\frac{d^4k}{(2\pi)^4}\frac{1}{2e^2}K_\mu(k)D^{\mu\nu}(k)K_\nu(-k)]\times$$
$$\exp[i\int\frac{d^4k}{(2\pi)^4}\bar{\Psi}(k)(\gamma^\mu k_\mu - m)\Psi(-k) - i\int\frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}K_\mu(k)\bar{\Psi}(p)\gamma^\mu\Psi(-k+p)] =$$
$$= \text{const}\int\prod_i dK_\mu(p_i)(\det[\frac{1}{2}(k^2g^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu)])^{-1/2}(\det[\frac{2i}{e^2}D_{\mu\nu}])^{1/2}\times$$
$$\exp[i\int\frac{d^4k}{(2\pi)^4}\frac{1}{2e^2}K_\mu(k)D^{\mu\nu}(k)K_\nu(-k)]\times\det[(\gamma^\mu k_\mu - m)\delta_{m,n} - \gamma^\mu(K_\mu)_{-m-n=k}] \tag{1.93}$$

Note that the result in Eq. (1.93) is exactly that of QED with the field A_μ replaced by the new variable K_μ. It seems that our derivation although correct is redundant. However in the next sections we will show that this procedure in its intermediate steps allows us to extract the corrections to the beta function for the electric charge in a new simplified renormalization scheme.

We write the expression for the two point function in the Fourier space:

$$\int\prod_i\prod_j\prod_k dA_\mu(p_i)d\bar{\Psi}(p_j)d\Psi(p_k)A_\rho(p)A_\sigma(q)\exp[i\int d^4x\mathcal{L}_{QED}] \tag{1.94}$$

We perform the same change of variable as in the first section $A_\nu(k) \to A_\nu(k) - \frac{e}{2}\Psi(p)\frac{\gamma^\nu}{D_{\mu\nu}}\Psi(p-k)$ to obtain:

$$
\begin{aligned}
I_{\rho\sigma} = &\int \prod_i \prod_j \prod_k dA_\mu(p_i) d\bar{\Psi}(p_j) d\Psi(p_k) \times \\
&[A_\rho(p)A_\sigma(q) + \frac{e^2}{4}\bar{\Psi}(r)\frac{\gamma^\mu}{D_{\mu\rho}(p)}\Psi(r+p)\bar{\Psi}(u)\frac{\gamma^\nu}{D_{\nu\sigma}(q)}\Psi(u+q)] \times \\
&\exp[i\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A_\mu(k) D_{\mu\nu} A_\nu(-k) + \int \frac{d^4k}{(2\pi)^4}\bar{\Psi}(k)(\gamma^\mu k_\mu - m)\Psi(-k) - \\
&\int e^2 \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}[\frac{1}{4}\bar{\Psi}(p)\gamma^\mu\Psi(p+k)D^{-1}_{\mu\nu}\bar{\Psi}(q)\gamma^\nu\Psi(q-k)]],
\end{aligned}
\tag{1.95}
$$

where this time the variable A_μ is the new variable and we dropped the odd terms that lead to zero in the functional integration.

Since the integrals over the gauge and fermion fields are independent one can write:

$$
\begin{aligned}
I_{\rho\sigma} = &\int \prod_i dA_\mu(p_i)[A_\rho(p)A_\sigma(q)]\exp[i\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A_\mu(k) D_{\mu\nu} A_\nu(-k)] \times \\
&\int \prod_j \prod_k d\bar{\Psi}(p_j) d\Psi(p_j) \exp[i\int \frac{d^4k}{(2\pi)^4}\bar{\Psi}(k)(\gamma^\mu k_\mu - m)\Psi(-k) - \\
&\int e^2 \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}[\frac{1}{2}\bar{\Psi}(p)\gamma^\mu\Psi(p+k)D^{-1}_{\mu\nu}(k)\bar{\Psi}(q)\gamma^\nu\Psi(q-k)]] + \\
&\int \prod_i dA_\mu(p_i) \exp[i\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A_\mu(k) D_{\mu\nu} A_\nu(-k)] \times \\
&\int \prod_j \prod_k d\bar{\Psi}(p_j) d\Psi(p_k)[\frac{e^2}{4}\bar{\Psi}(r)\frac{\gamma^\mu}{D_{\mu\rho}(p)}\Psi(r+p)\bar{\Psi}(u)\frac{\gamma^\nu}{D_{\nu\sigma}(q)}\Psi(u+q)] \times \\
&\exp[i\int \frac{d^4k}{(2\pi)^4}\bar{\Psi}(k)(\gamma^\mu k_\mu - m)\Psi(-k) - \\
&\int e^2 \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}[\frac{1}{2}\bar{\Psi}(p)\gamma^\mu\Psi(p+k)D^{-1}_{\mu\nu}\bar{\Psi}(q)\gamma^\nu\Psi(q-k)]]
\end{aligned}
\tag{1.96}
$$

Then the first term in Eq. (1.96) can be separated and leads to upon functional integration:

$$
\begin{aligned}
&\int \prod_i dA_\mu(p_i)[A^\rho(k)A^\sigma(q)]\exp[i\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} A_\mu(k) D_{\mu\nu} A_\nu(-k)] = \\
&\delta(k-q)\frac{i\xi}{2}\frac{\delta^2}{\delta k_\rho \delta k_\sigma}(\det[i(-k^2 g^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu])^{-1} = \\
&\frac{-i}{k^2}(g^{\rho\sigma} - \frac{k^\rho k^\sigma}{k^2})(\det[i(-k^2 g^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu])^{-1},
\end{aligned}
\tag{1.97}
$$

which corresponds to the free field propagator for the gauge field in the Landau gauge.

First note that in the functional integration over a Fourier mode the mode in question should be regarded as an independent variable and the various momenta that appear in the Lagrangian as independent parameters. Then an operator like $\frac{i\xi}{2}\frac{\delta^2}{\delta k_\rho \delta k_\sigma}$ which is a function only of the parameters acts only on the parameters and does not affect in any way the variable of integration. This is the correct procedure in the path integral formalism.

Eq. (1.97) needs further clarifications. Second we note that the quadratic kinetic operator that appears in the Lagrangian is singular such that we need to introduce the gauge parameter for consistency. We shall conduct our calculations in specific gauge with $\xi = 0$. Then,

$$\begin{aligned} & i\xi\frac{\partial^2}{\partial k_\rho k_\sigma} i(-k^2 A_\mu(k)A^\mu(-k) + (1-\frac{1}{\xi})k^\mu k^\nu A_\mu(k)A_\nu(-k)) = \\ & \xi[-2g^{\rho\sigma}A_\mu(k)A^\mu(-k) + 2(1-\frac{1}{\xi})A^\rho(k)A^\sigma(-k)] = A^\rho(k)A^\sigma(-k) \end{aligned} \tag{1.98}$$

in the limit $\xi = 0$. Thus our operator is adjusted for the Landau gauge and for every chosen gauge parameter one should associate a different operator.

In the same context let us note that our operator $D_{\mu\nu}$ satisfies the equation:

$$[-k^2 g^{\mu\nu} + (1-\frac{1}{\xi})k^\mu k^\nu]\tilde{D}^{\nu\rho} = i\delta^\rho_\mu, \tag{1.99}$$

where $\tilde{D}^{\nu\rho}$ is the inverse operator:

$$\tilde{D}^{\nu\rho} = \frac{-i}{k^2}(g^{\nu\rho} - (1-\xi)\frac{k^\nu k^\rho}{k^2}). \tag{1.100}$$

Then one can infer:

$$\frac{\xi}{2}\frac{\delta^2}{\delta k^\rho \delta k^\sigma}\frac{1}{D_{\mu\nu}} = \frac{1}{D_{\mu\sigma}D_{\nu\rho}}. \tag{1.101}$$

again in the same Landau gauge (Eq. (1.101) is not universal as it can be checked easily). Here the derivative contains also other terms that we ignore because do not correspond to one particle irreducible diagrams.

We then claim that:

$$\int \prod_i \prod_j d\bar{\Psi}(p_i) d\Psi(p_j) [\frac{e^2}{4} \bar{\Psi}(r) \frac{\gamma^\mu}{D_{\mu\rho}(p)} \Psi(r+p) \bar{\Psi}(s) \frac{\gamma^\nu}{D_{\nu\sigma}(q)} \Psi(s+q)] \times$$
$$\exp\left[i \int \frac{d^4k}{(2\pi)^4} \bar{\Psi}(k)(\gamma^\mu k_\mu - m)\Psi(-k) - \right.$$
$$\int e^2 \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} [\frac{1}{2} \bar{\Psi}(p)\gamma^\mu \Psi(p+k) D^{-1}_{\mu\nu} \bar{\Psi}(q) \gamma^\nu \Psi(q-k)]] =$$
$$\frac{i\xi}{2} \frac{\partial^2}{\partial k^\rho k^\sigma} \int \prod_i \prod_j d\bar{\Psi}(p_i) d\Psi(p_j) \exp[i \int \frac{d^4k}{(2\pi)^4} \bar{\Psi}(k)(\gamma^\mu k_\mu - m)\Psi(-k) -$$
$$\left. \int e^2 \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} [\frac{1}{2} \bar{\Psi}(p)\gamma^\mu \Psi(p+k) D^{-1}_{\mu\nu} \bar{\Psi}(q) \gamma^\nu \Psi(q-k)] \right] \tag{1.102}$$

Now if we trace back from Eq. (1.93) the contribution from Eq. (1.102) we see that we need to consider the quantity:

$$\frac{i\xi}{2} \frac{\partial^2}{\partial k^\rho \partial k^\sigma} \text{const} \int \prod_i dK_\mu(p_i) (\det[\frac{2i}{e^2} D_{\mu\nu}])^{1/2} (\det[iD_{\mu\nu}])^{-1/2} \times$$
$$\exp[i \int \frac{d^4k}{(2\pi)^4} \frac{1}{2e^2} K_\mu(k) D^{\mu\nu} K_\nu(-k)] \det[(\gamma^\mu p_\mu - m)\delta_{mn} - \gamma^\mu (K_\mu)_{-m-n=k}] =$$
$$(\det[\frac{2}{e^2}])^{1/2} \frac{i\xi}{2} \frac{\partial^2}{\partial k^\rho \partial k^\sigma} \exp[i \int \frac{d^4k}{(2\pi)^4} \frac{1}{2e^2} K_\mu(k) D^{\mu\nu}(k) K_\nu(-k)] \times$$
$$\det[(\gamma^\mu p_\mu - m)\delta_{mn} - \gamma^\mu (K_\mu)_{-m-n=k}], \tag{1.103}$$

as it is evident that the two factors cancel each other. We then make the change of variable $\frac{K_\mu}{e} \to K_\mu$ to obtain:

$$I_{\rho\sigma} = \text{const} \frac{i\xi}{2} \frac{\partial^2}{\partial k^\rho \partial k^\sigma} \int dK_\mu \exp[i \int \frac{d^4k}{(2\pi)^4} \frac{1}{2} K_\mu(k) D^{\mu\nu} K_\nu(-k)] \times$$
$$\det[(\gamma^\mu p_\mu - m)\delta_{mn} - e\gamma^\mu (K_\mu)_{-m-n=k}] \tag{1.104}$$

In order to extract the corrections to the charge renormalization we need to make a new change of variable. The operator,

$$O_{\alpha\mu} = \sqrt{\frac{i}{2k^2}} [k^2 g^{\mu\alpha} - (1 - \frac{1}{\sqrt{\xi}}) k^\mu k^\alpha] \tag{1.105}$$

satisfies the relation,

$$O_{\alpha\mu} O^\alpha_\nu = -\frac{i}{2} D_{\mu\nu} \tag{1.106}$$

where $D_{\mu\nu}$ is given in Eq. (1.90). We then make the change of variable:

$$K'_{\mu} = K_{\alpha} O^{\alpha}_{\mu}. \tag{1.107}$$

For simplicity we rename the new variable also K_{μ}. This leads to :

$$\begin{aligned}
&I_{\rho\sigma} = \text{const}\frac{i\xi}{2}\frac{\partial^2}{\partial k^{\rho}\partial k^{\sigma}}\int\prod_i dK_{\mu}(p_i)(\det[i(k^2 g^{\mu\nu} - (1-\frac{1}{\xi})k^{\mu}k^{\nu}])^{-1} \times \\
&\exp[-\int\frac{d^4k}{(2\pi)^4}\frac{1}{2}K_{\mu}(k)K_{\nu}(-k)] \times \\
&\det[(\gamma^{\mu}p_{\mu} - m)\delta_{mn} - \gamma^{\mu}e(K_{\alpha})_{-m-n=k}(O^{\alpha}_{\mu})^{-1}].
\end{aligned} \tag{1.108}$$

The first contribution of the derivative is coming from:

$$\begin{aligned}
&\frac{i\xi}{2}\frac{\partial^2}{\partial k^{\rho}\partial k^{\sigma}}\det[k^2 g^{\mu\nu} - (1-\frac{1}{\xi})k^{\mu}k^{\nu}])^{-1} = \\
&-i\frac{1}{k^2}(g^{\rho\sigma} - (1-\xi)\frac{k^{\rho}k^{\sigma}}{k^2})\det[i(k^2 g^{\nu\mu} - (1-\frac{1}{\xi})k^{\mu}k^{\nu})])^{-1}
\end{aligned} \tag{1.109}$$

which is the free field propagator. Note that this is actually the contribution that we had obtained in Eq. (1.97) as we applied again the operator $\frac{i\xi}{2}\frac{\partial^2}{\partial k^{\rho}\partial k^{\sigma}}$ to the full partition function.

Then the first extra contributions come from term of the type:

$$\frac{\partial}{\partial k^{\rho}}(\det[k^2 g^{\mu\nu} - (1-\frac{1}{\xi})k^{\mu}k^{\nu}])^{-1}\frac{\partial}{\partial k^{\sigma}}\det[(\gamma^{\mu}p_{\mu} - m)\delta_{mn} - e\gamma^{\mu}(K_{\alpha})_{-m-n=k}(O^{\alpha}_{\mu})^{-1}], \tag{1.110}$$

which we claim is zero. To prove that let us consider the first factor in Eq. (1.110):

$$\frac{\delta}{\delta k^{\rho}}\det[k^2 g^{\mu\nu} - (1-\frac{1}{\xi})k^{\mu}k^{\nu}] = -8k^{\rho}\det[k^2 g^{\mu\nu} - (1-\frac{1}{\xi})k^{\mu}k^{\nu}]. \tag{1.111}$$

In order to make the point we first outline our next steps. These comprise the calculation of the derivatives of the various K_{μ} terms. After we perform these we will change again the variable to $K_{\alpha} \to -2iO^{\mu}_{\alpha}K_{\mu}$. It turns out that one derivative of the type $\frac{\delta}{\delta k^{\rho}}$ applied to a K_{μ} term and after the new change of variable mentioned leads to terms which contain at most the inverse power $\frac{1}{\sqrt{\xi}}$. Then the result in Eq. (1.111) together with the result of differentiating the K_{μ} term will contain at most terms of type $\frac{1}{\sqrt{\xi}}$ which multiplied by ξ in the limit $\xi = 0$ leads to zero.

Second reason why we should not consider the term in Eq. (1.110) is that this represents some first order correction. We are interested in computing in this approach the beta function and for all purposes we can consider for that:

$$-ig_{\mu\nu}\frac{1}{k^2(1-\Pi(k^2))}. \tag{1.112}$$

such that the terms proportional to $k^\rho k^\sigma$ can be neglected. We shall do that for the rest of calculations for the sake of simplicity.

Moreover since we are actually interested in computing the beta function we need $\Pi(0)$ so we can safely take the limit $k^2 = 0$ in the correction. In this context the variables $K_\mu(p)$ with $p^2 \neq k^2$ that appear in the determinant can be considered as having a zero contribution and thus be neglected.

Next we need to determine the contributions proportional to:

$$\frac{i\xi}{2}\frac{\partial^2}{\partial k^\rho \partial k^\sigma}\det[(\gamma^\mu p_\mu - m)\delta_{mn} - e\gamma^\mu (K_\alpha)_{-m-n=k}(O^\alpha_\mu)^{-1}] \tag{1.113}$$

We need to consider the formula of differentiation of a determinant.

$$\begin{aligned}\frac{1}{2}\frac{\partial^2}{\partial k^\rho \partial k^\sigma}\det A = \frac{1}{2}\det A\mathrm{Tr}[\frac{\partial A}{\partial k^\rho}A^{-1}]\mathrm{Tr}[\frac{\partial A}{\partial k^\sigma}A^{-1}] +\\ \frac{1}{2}\det A\mathrm{Tr}[\frac{\partial^2 A}{\partial k^\rho \partial k^\sigma}A^{-1}] - \frac{1}{2}\det A\mathrm{Tr}[\frac{\partial A}{\partial k^\rho}A^{-1}\frac{\partial A}{\partial k^\sigma}A^{-1}].\end{aligned} \tag{1.114}$$

where,

$$A = [(\gamma^\mu p_\mu - m)\delta_{mn} - e\gamma^\mu (K_\alpha)_{-m-n=k}(O^\alpha_\mu)^{-1}]. \tag{1.115}$$

Here we should note that the term that contains $\frac{\partial^2 A}{\partial k^\rho k^\sigma}$ is zero by the same reasons as the term in Eq. (1.110).

In order to determine the charge renormalization given by $\frac{1}{1-\Pi(0)}$ and in consequence the beta function in this method of functional integration we need to determine two terms:

$$\begin{aligned}-i\xi\frac{1}{2}\det A\mathrm{Tr}[\frac{\partial A}{\partial k^\rho}A^{-1}\frac{\partial A}{\partial k^\sigma}A^{-1}]\\ i\xi\frac{1}{2}\det A\mathrm{Tr}[\frac{\partial A}{\partial k^\rho}A^{-1}]\mathrm{Tr}[\frac{\partial A}{\partial k^\sigma}A^{-1}],\end{aligned} \tag{1.116}$$

where the first term corresponds to the one loop contribution and the second to the two loops one. The beta function stops at two loops. This stems from the number of integral over the momentum. Note that A^{-1} actually contains an expansion in $(e^2)^n$ where n can go to infinity. However the corresponding terms do not increase the number of loops but lead to an increasing number of external legs in the one loop and two loop corrections. Since we are interested only in the two point functions such terms should be dropped.

In order to determine terms of the type $\frac{\delta A}{\delta k^\rho}$ we note that the presence of the factor ξ in front means that we need to compute only the contribution proportional to $\frac{1}{\xi}$. Thus,

$$\frac{\delta A}{\delta k_\rho} = eK_\alpha\sqrt{\frac{2k^2}{i}}(1-\sqrt{\xi})\frac{1}{k^2}(g^{\mu\rho}\frac{k^\alpha}{k^2} + g^{\alpha\rho}\frac{k^\mu}{k^2}) \tag{1.117}$$

Then we make the new change of variable: $K^\alpha = K'_\nu \frac{k^2 g^{\nu\alpha} - (1-\frac{1}{\sqrt{\xi}}) k^\nu k^\alpha}{\sqrt{k^2}}$ to find:

$$eK_\alpha \sqrt{\frac{2k^2}{i}}(1-\sqrt{\xi})\frac{1}{k^2}(g^{\mu\rho}\frac{k^\alpha}{k^2} + g^{\alpha\rho}\frac{k^\mu}{k^2}) \rightarrow e\sqrt{\frac{2}{i}}\frac{1}{k^2}\frac{1}{\sqrt{\xi}}K'_\nu k^\nu g^{\mu\rho} \qquad (1.118)$$

where only the term proportional to $\frac{1}{\sqrt{\xi}}$ is retained.

We will show in some detail only how the first contribution in Eq. (1.120) is obtained. Thus,

$$\begin{aligned}
&-i\xi\frac{1}{2}\det A \mathrm{Tr}[\frac{\partial A}{\partial k^\rho}A^{-1}\frac{\partial A}{\partial k^\sigma}A^{-1}] = \\
&-\frac{i}{2(k^2)^2}\frac{2}{i}\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-m^2)^2}\mathrm{Tr}e^2[k^\alpha K_\alpha k^\beta K_\beta \gamma^\rho(\gamma^\tau p_\tau + m)\gamma^\sigma(\gamma^\eta p_\eta + m)] = \\
&= -ig^{\rho\sigma}\frac{1}{k^2}\frac{2}{64\pi^2}[-\Lambda^2 + 4m^2\ln[\frac{\Lambda^2}{m^2}]]K_\mu^2 e^2 \qquad (1.119)
\end{aligned}$$

We are forced to use a cut-off procedure along with an Euclidean space for both momenta and the field $K(k^2=0)$ that we denoted simply by K_μ. Note that the actual result gets divided by the zeroth order partition function and all other factors dependent on the other $K_\mu(q)$ variables get canceled. Thus the result can be written as:

$$-e^2\frac{2}{64\pi^2}[\Lambda^2 - 4m^2\ln[\frac{\Lambda^2}{m^2}]]\frac{\int dK_\mu K_\mu^2}{\int dK_\mu} = -e^2\frac{1}{16\pi^2}\frac{1}{3}[1-4\frac{m^2}{\Lambda^2}\ln[\frac{\Lambda^2}{m^2}]], \qquad (1.120)$$

where the integral over K is performed in spherical coordinates in the four dimensional Euclidean space with a cut-off Λ. Here we need to note that the actual $K_\mu(p)$ coordinate has dimension of m^{-3} but since it comes with an integral d^4p one includes this factor in the variable to get a dimension of m. Then when one consider the traces one needs to include a factor of $\frac{1}{V}$ to go from summation to integration. This multiplies the result by an extra factor of V which is equivalent to dividing by $\frac{1}{\Lambda^4}$.

The two loop term is calculated as easily:

$$\begin{aligned}
&i\xi\frac{1}{2}\det A\mathrm{Tr}[\frac{\partial A}{\partial k^\rho}A^{-1}]\mathrm{Tr}[\frac{\partial A}{\partial k^\sigma}A^{-1}] = \\
&ig^{\rho\sigma}\frac{1}{k^2}\frac{1}{3}\frac{1}{64\pi^4}(-\Lambda^2 + 3m^2\ln[\frac{\Lambda^2}{m^2}])^2(K_\mu^2)^2 \qquad (1.121)
\end{aligned}$$

This again needs to be divided by the zeroth order partition function which leads to:

$$-e^4\frac{1}{64\pi^4}(-\Lambda^2 + 3m^2\ln[\frac{\Lambda^2}{m^2}])^2\frac{\int dK_\mu(K_\mu^2)^2}{\int dK_\mu} = -e^4\frac{1}{64\pi^4}(-1+3\frac{m^2}{\Lambda^2}\ln[\frac{\Lambda^2}{m^2}])^2\frac{1}{2}. \quad (1.122)$$

Before going further we need to explain why Eqs. (1.120) and (1.122) do not contain any trace of the determinant and of the exponential present initially in Eq. (1.104). For that we note that upon the expansion the determinant will be a polynomial of order $N \to \infty$ in the modes $K_\mu(p)$. We arrange this polynomial in terms of the mode $K_\mu(q)$ with $q^2 = m^2 + i\epsilon$. The corresponding polynomial multiplied by gaussian integrals is then solved to lead to a new polynomial with fractional powers in terms of the quantities $\frac{1}{q^2-m^2+i\epsilon}$. The other terms in the expansion of the determinant $K_\mu(p)$ with $p \neq q$ and $p^2 \neq 0$ do not contribute as they are multiplied by the small parameter ϵ. Let us see how this works by considering an integral of only two variables (where $b = i\epsilon$):

$$\int dXdY[X^{2N} + Y^N + a_1 X^{2N-1} + b_1 Y X^{2N-1} + ...]\exp[-bX^2]Y^2 =$$
$$\mathrm{const}\frac{1}{\epsilon^{1/2+N}}\int dYY^2. \tag{1.123}$$

Here X corresponds to the mode $K_\mu(q)$ whereas Y corresponds to $K_\mu(k)$ that is integrated in the Eqs. (1.120) and (1.122) and we factorized the result of integration for X^{2N} and took the limit $\epsilon = 0$ in the rest. The factor $\frac{1}{\epsilon^{1/2+N}}$ gets canceled by the corresponding term in the denominator. Thus this justifies the results in Eqs. (1.120) and (1.122).

By adding up the results in Eqs. (1.162) and (1.122) we obtain for the full correction to the charge renormalization the quantity:

$$\Pi(0) = e^2\frac{1}{16\pi^2}\frac{1}{3}[-1+\frac{4m^2}{\Lambda^2}\ln[\frac{\Lambda^2}{m^2}]] - e^4\frac{1}{128\pi^4}(\frac{1}{3} - 2\frac{m^2}{\Lambda^2}\ln[\frac{\Lambda^2}{m^2}] + 3\frac{m^4}{\Lambda^4}(\ln[\frac{\Lambda^2}{m^2}])^2] \tag{1.124}$$

Note that here Λ is just an intermediate scale and one cannot take the limit $\Lambda \to \infty$ as this does not make sense. The structure that we obtain is not unusual since one expects that a naive cut-off procedure violates the Ward identity and leads to an infinite photon mass. However when we integrated only over the modes K_μ with $k^2 = 0$ we made the underlying assumption that the lower modes are relevant such that the cut-off should not be too high. Also one cannot take the cut-off so low as the order of m because then the computation of the beta function does not make sense. We thus will consider $\Lambda > m$ but not $\Lambda \gg m$. In this context it make sense to use the expansion $\frac{m^2}{\Lambda^2} = \exp[-\ln[\frac{\Lambda^2}{m^2}]] = 1 - \ln[\frac{\Lambda^2}{m^2}] + ...$ We claim that since this procedure contains implicitly the higher order loops it is natural to have higher power of logarithms even at two loops.

Finally one can compute from Eq. (1.124) and obtain for the full beta function:

$$\beta(\alpha) = \frac{\partial(\frac{\alpha}{\pi})}{\partial \ln[M^2]} = \frac{1}{3}(\frac{\alpha}{\pi})^2 + \frac{1}{4}(\frac{\alpha}{\pi})^4, \tag{1.125}$$

which agrees with the standard result for the two coefficients of the beta function that are renormalization scheme independent. (Here we replaced the cut-off scale by a renormalization scale M).

There are no higher corrections in this procedure to the result in Eq. (1.125).

In the end we will show that this result does not contradict any of the known facts in perturbation theory. We consider the invariant charge defined as:

$$e_d^2(q^2) = \frac{e_0^2}{1 - \Pi(q^2, \Lambda^2, e_0^2, m^2)}. \tag{1.126}$$

This charge depends on the momenta and can be written in an RG invariant form. However when one computes $\Pi(q^2)$ with $q^2 \neq 0$ the arguments we gave in the paragraph after Eq. (36) do not apply anymore. Our calculations are valid for this case up to the Eq. (32). From that point on one cannot ignore the exponential and the determinant that appear in Eq. (18) and in order to perform the corresponding calculations one should use the formula of expansion for a determinant:

$$\det(1 + A) = \exp[\sum_{n=1}^{\infty} -\frac{(-1)^n}{n} \mathrm{Tr} A^n] \tag{1.127}$$

If one further expands the exponential in Eq. (1.127) one obtains traces at any powers. Since each trace is equivalent with the integral over one momentum the number of traces correspond to the number of loops or to the order in perturbation theory. Thus on general grounds the invariant charge will contain corrections of higher orders than two. Considering the above discussion one can write:

$$\begin{aligned}
&\frac{de_d^2}{d\ln\Lambda^2} = \beta_0 e_d^4 + \beta_1 e_d^6 + \beta_2 e_d^8 + ...\\
&\frac{de^2}{d\ln\Lambda^2} = \beta_0 e^4 + \beta_1 e^6\\
&e_d^2 = f(e^2) = e^2 + \alpha_1 e^4 + \alpha_2 e^6 + \alpha_3 e^8 + ...
\end{aligned} \tag{1.128}$$

and thus obtain $\alpha_2 = \beta_2/\beta_0$ etc. This shows that indeed higher order corrections are manifest in the expansion of the invariant charge.

D. Partition function for a gauge theory with strong interaction in a new functional approach.

In the previous sections we introduced nonperturbative methods for computing an all order correction to the mass of the scalar in the Φ^4 theory and also we computed in a semiperturbative method the beta function for QED with fermions in the fundamental representation. Unfortunately although the methods were based on the same idea for each case we were forced to introduce new techniques thus making the approach difficult to use for a general case. In this work we introduce a new method for the Yang Mills theory which can be applied for any renormalizable field theory directly. Instead of relying on a perturbative approach our purpose was to determine global properties of the theory from the specific properties of the zero current partition function. We rely on the path integral formalism to obtain useful relations between the renormalization constants of the Yang Mills theory. These relations lead for the background gauge field method to a derivation of the general form of the beta function. It turns out that through this method one obtains an all loop beta function with only the first two coefficients different than zero thus indicating that the 't Hooft scheme might be in a sense the most natural scheme for beta functions.

We start with the Yang Mills Lagrangian:

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2, \tag{1.129}$$

where,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{1.130}$$

The Lagrangian in Eq. (1.167) needs fixing. This is done by introducing the ghost Lagrangian:

$$\mathcal{L}_g = \bar{c}^a(-\partial^\mu\partial_\mu - g f^{abc}\partial^\mu A^b_\mu)c^c. \tag{1.131}$$

We shall work in the Feynman gauge ($\xi = 1$) and in the Fourier space throughout this paper. Thus starting from,

$$A^a_\mu(x) = \frac{1}{V}\sum_n \exp[-ik_n x] A^a_\mu(k_n) \tag{1.132}$$

we rewrite the full Lagrangian:

$$\begin{aligned}
&\int d^4x\mathcal{L} = -\frac{1}{2}\frac{1}{V}\sum_n k_n^2 A^{a\nu}(k_n)A^a_\nu(-k_n) + \frac{1}{V}\sum_n k_n^2 \bar{c}^a(k_n)c^a(-k_n) + \\
&+\frac{i}{V^2}\sum_{n,m} k_n^\mu A^a_\nu(k_n) f^{abc} A^b_\mu(k_m) A^{c\nu}(-k_n-k_m) - \\
&-\frac{1}{V^3} f^{abc} f^{ade} \sum_{n,m,p} A^{b\mu}(k_n) A^{c\nu}(k_m) A^d_\mu(k_p) A^e_\nu(-k_n-k_m-k_p) - \\
&-\frac{i}{V^2}\sum_{n,m} k_n^\mu \bar{c}^a(k_n) g f^{abc} A^b_\mu(k_m) c^c(-k_n-k_m).
\end{aligned} \tag{1.133}$$

Then one defines the zero current partition function by the expression:

$$Z_0 = \int \prod_i \prod_j \prod_m dA_\mu^a(k_i) d\bar{c}^b(k_j) dc^d(k_m) \exp[i \int d^4x \mathcal{L}], \tag{1.134}$$

where in the exponent one should use the Eq. (1.171).

It is useful at this stage to settle some of the properties of Z_0. It is known that Z_0 apart from factor in front is given by the exponential of the sum of all disconnected diagrams:

$$Z_0 = \text{factor} \times \exp[\sum_i V_i] \tag{1.135}$$

where V_i is a typical disconnected diagram. Since the calculation is done in the absence of external sources all V_i diagrams are closed and contain summations over momenta (that appear in propagators or vertices) and thus do not depend at all on any momenta. The factor in front is a product obtained form integrating the gaussian integrals corresponding to the kinetic terms. So one can write:

$$Z_0 = \text{const} \prod_i (k_i^2)^{N^2-1} \prod_j (k_j^2)^{-d/2(N^2-1)} \exp[\sum_i V_i] \tag{1.136}$$

where N is coming from the Yang Mills group $SU(N)$ and the first factor corresponds to the ghosts whereas the second to the gluon fields.

We write:

$$\begin{aligned}
Z_0 = & \int \prod_i \prod_j \prod_m dA_\mu^a(k_i) d\bar{c}^b(k_j) dc^d(k_m) \exp[i \int d^4x \mathcal{L}] = \\
= & \int \prod_i \prod_j \prod_m dA_\mu^a(k_i) d\bar{c}^b(k_j) dc^d(k_m) \frac{dA_\nu^a(k)}{dA_\nu^a(k)} \exp[i \int d^4x \mathcal{L}] = \\
= & \int \prod_i \prod_j \prod_m dA_\mu^a(k_i) d\bar{c}^b(k_j) dc^d(k_m) \frac{d}{dA_\nu^a(k)} [A_\nu^a(k) \exp[i \int d^4x \mathcal{L}]] - \\
- & \int \prod_i \prod_j \prod_m dA_\mu^a(k_i) d\bar{c}^b(k_j) dc^d(k_m) A_\nu^a(k) \frac{d}{dA_\nu^a(k)} \exp[i \int d^4x \mathcal{L}].
\end{aligned} \tag{1.137}$$

We start by analyzing the first term on the right side of the Eq. (1.175) to get:

$$\begin{aligned}
& \int \prod_i \prod_j \prod_m dA_\mu^a(k_i) d\bar{c}^b(k_j) dc^d(k_m) A_\nu^a(k) \exp[i \int d^4x \mathcal{L}]_{A_\nu^a(k)=+\infty} - \\
& \int \prod_i \prod_j \prod_m dA_\mu^a(k_i) d\bar{c}^b(k_j) dc^d(k_m) A_\nu^a(k) \exp[i \int d^4x \mathcal{L}]_{A_\nu^a(k)=-\infty}.
\end{aligned} \tag{1.138}$$

Although the Fourier transform of the gauge field has a real and a imaginary part we can assume that it is real without any loss of generality as the same arguments apply. We first note that the exponential factor in Eq. (1.176) will contain:

$$\exp[i\int d^4x\mathcal{L}] \sim \text{other factors} \times \exp[-\frac{i}{2}k^2 A^{a\nu}(k)A^a_\nu(k)] \tag{1.139}$$

However k^2 should actually be written as $k^2 + i\epsilon$ where ϵ ensures the convergence of the gaussian integral corresponding to the term in Eq. (1.139). Then the limits in Eq. (1.176) will be zero as they contain an exponential that goes to zero as it can be seen from :

$$\lim_{A^a_\nu \to \pm\infty} A^a_\nu(k)\exp[-\frac{i}{2}k^2 A^{a\nu}(k)A^a_\nu(k) - \frac{\epsilon}{2}A_{a\nu}(k)A^a_\nu(k)] = 0 \tag{1.140}$$

Note that we picked a space time component ν such that the corresponding metric for it is $g^{\nu\nu} = -1$. Thus the first contribution on the right hand side of the Eq. (1.175) cancels. The second contribution is given by:

$$\begin{aligned} Z_0 = &\int \prod_i \prod_j \prod_m dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)(-i)[-\frac{k^2}{V}A^{a\nu}(k)A^a_\nu(-k) + \\ &\frac{3i}{V^2}gk^\mu \sum_p f^{abc}A^a_\nu(k)A^b_\mu(p)A^{c\nu}(-k-p) - \frac{i}{V^2}g\sum_p p^\nu\bar{c}^b(p)f^{bac}A^a_\nu(k)c^c(-p-k) - \\ &-\frac{1}{V^3}g^2 f^{bac}f^{bde}\sum_{p,q} A^a_\nu(k)A^c_\mu(p)A^{d\nu}(q)A^{e\mu}(-p-k-q)] \times \exp[i\int d^4x\mathcal{L}]. \end{aligned} \tag{1.141}$$

According to Eq. (1.238) one can write:

$$k^\mu \frac{dZ_0}{\partial k^\mu} = -2(N^2-1)[\frac{d}{2}-1]Z_0 \tag{1.142}$$

We further apply the operator $k^\mu \frac{d}{dk^\mu}$ to Eq. (1.228) to obtain:

$$\begin{aligned} k^\mu \frac{dZ_0}{dk^\mu} = &\int \prod_i \prod_j \prod_m dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m) \times \\ &i[-\frac{1}{V}k^2 A^{a\nu}(k)A^a_\nu(-k) + \frac{2}{V}k^2\bar{c}^a(k)c^a(-k) + \frac{i}{V^2}k^\mu\sum_p A^a_\nu(k)f^{abc}gA^b_\mu(p)A^{c\nu}(-p-k) - \\ &-\frac{i}{V^2}\sum_p k^\mu\bar{c}^a(k)gf^{abc}A^b_\mu(p)c^c(-p-k)] \times \exp[i\int d^4x\mathcal{L}. \end{aligned} \tag{1.143}$$

The next step will be to reconsider the Lagrangian from the perspective of renormalization. Thus the fields are rescaled using the standard procedure which leads to:

$$\mathcal{L}_r = -\frac{1}{2}\frac{1}{V}Z_3\sum_n k_n^2 A^{a\nu}(k_n)A^a_\nu(-k_n) + \frac{1}{V}Z_2\sum_n k_n^2\bar{c}^a(k_n)c^a(-k_n) +$$
$$+\frac{i}{V^2}Z_{3g}\sum_{n,m}k_n^\mu A^a_\nu(k_n)f^{abc}A^b_\mu(k_m)A^{c\nu}(-k_n-k_m) -$$
$$-\frac{1}{V^3}Z_{4g}g^2 f^{abc}f^{ade}\sum_{n,m,p}A^{b\mu}(k_n)A^{c\nu}(k_m)A^d_\mu(k_p)A^e_\nu(-k_n-k_m-k_p) -$$
$$-\frac{i}{V^2}Z_1'\sum_{n,m}k_n^\mu\bar{c}^a(k_n)gf^{abc}A^b_\mu(k_m)c^c(-k_n-k_m). \tag{1.144}$$

Here the fields and the couplings should be considered the renormalized ones and the renormalization constants satisfy the Slanov-Taylor identities:

$$g_0^2 = \frac{Z_{3g}^2}{Z_3^3}g^2\mu^\epsilon = \frac{Z_{4g}}{Z_3^2}g^2\mu^\epsilon = \frac{Z_1'^2}{Z_2'^2 Z_3}g^2\mu^\epsilon, \tag{1.145}$$

where $d = 4-\epsilon$ and μ is a parameter with dimension of mass and we shall use dimensional regularization scheme.

As an aside note that in the background gauge field method which consists in the separation of the gauge field A^a_μ into a background gauge field B^a_μ and a quantum fluctuation $\tilde{A}^a_\mu$ only the background gauge field gets renormalized as the quantum fluctuations appear only inside loops and one has in this case a simple relation among the renormalization constants:

$$Z_{4g} = Z_{3g} = Z_3$$
$$Z_2 = Z_1'$$
$$Z_g = Z_3^{-1/2}. \tag{1.146}$$

Now consider that instead of applying the procedure that led to the Eqs. (1.177) and (1.181) to the bare Lagrangian we apply it to the renormalized one. Then Eqs.(1.177) and (1.181) will become:

$$Z_0 = \int\prod_i\prod_j\prod_m dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)(-i)[-\frac{k^2}{V}Z_3A^{a\nu}(k)A^a_\nu(-k) +$$
$$\frac{3i}{V^2}Z_{3g}gk^\mu\sum_p f^{abc}A^a_\nu(k)A^b_\mu(p)A^{c\nu}(-k-p) - \frac{i}{V^2}gZ_1'\sum_p p^\nu\bar{c}^b(p)f^{bac}A^a_\nu(k)c^c(-p-k) -$$
$$-\frac{1}{V^3}g^2Z_{4g}f^{bac}f^{bde}\sum_{p,q}A^a_\nu(k)A^c_\mu(p)A^{d\nu}(q)A^{e\mu}(-p-k-q)]\times\exp[i\int d^4x\mathcal{L}], \tag{1.147}$$

and

$$-2(N^2-1)[\frac{d}{2}-1]Z_0 = \int \prod_i \prod_j \prod_m dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m) \times$$
$$i[-\frac{1}{V}Z_3k^2A^{a\nu}(k)A^a_\nu(-k) + \frac{2}{V}Z_2k^2\bar{c}^a(k)c^a(-k) + \frac{i}{V^2}k^\mu Z_{3g}\sum_p A^a_\nu(k)f^{abc}gA^b_\mu(p)A^{c\nu}(-p-k) -$$
$$-\frac{i}{V^2}Z_1'\sum_p k^\mu\bar{c}^a(k)gf^{abc}A^b_\mu(p)c^c(-p-k)] \times \exp[i\int d^4x\mathcal{L}]. \tag{1.148}$$

First let us review the equivalence that exist between the interaction picture in QFT and the path integral formalism. We illustrate this for the two point functions of a gauge theory theory although this is generally applicable:

$$\langle\Omega|T[A^a_\mu(x_1)A^b_\nu(x_2)]|\Omega\rangle =$$
$$\lim_{T\to(1-i\epsilon)}\frac{\langle 0|T[A^a_\mu(x_1)A^b_\nu(x_2)]\exp[-i\int_{-T}^{T}dtH_I(t)]|0\rangle}{\langle 0|\exp[-i\int^T --TdtH_I(t)]|0\rangle} =$$
$$\lim_{T\to(1-i\epsilon)}\frac{\int dA^c_\rho d\bar{c}^d dc^e A^a_\mu(x_1)A^b_\nu(x_2)\exp[i\int d^4x\mathcal{L}]}{\int dA^c_\rho\exp[i\int d^4x\mathcal{L}]} \tag{1.149}$$

To this we should add the known LSZ reduction formula which can be applied similarly to the interaction picture and to the path integral formalism:

$$\langle\Omega|T[A^a_\mu(p_1)...A^d_\nu(p_m)A^b_\rho(k_1)...A^e_\sigma(k_n)]|\Omega\rangle \sim$$
$$\sim_{p_i^0(k_j^0)\to E_{\vec{p}_i}(E_{\vec{k}_j})} \text{polarization factor} \times \text{const}\left(\prod_{i=1}^m \frac{iZ_3^{1/2}}{p_i^2+i\epsilon}\right)\left(\prod_{j=1}^n \frac{iZ_3^{1/2}}{k_j^2+i\epsilon}\right) \tag{1.150}$$

We shall apply the LSZ reduction formula following the equivalence given in Eq. (1.188) to Eqs. (1.147) and (1.148) to get:

$$1 = a_1Z_3 + a_2gZ_{3g}\sum_p k^\mu\frac{1}{k^2p^2(p+k)^2}f^{abc}\langle\vec{k},\epsilon_{a,\nu};\vec{p},\epsilon_{b,\mu}|S|\overrightarrow{p+k},\epsilon_{c,\nu}\rangle +$$
$$+a_2gZ_1'\sum p^\mu\frac{1}{p^2k^2(p+k)^2}f^{abc}\langle\vec{p},a;\vec{k},\epsilon_{b,\mu}|S|\overrightarrow{p+k},c\rangle +$$
$$+a_3g^2Z_{4g}f^{bac}f^{bde}\langle\vec{k},\epsilon_{a,\nu};\vec{p},\epsilon_{c,\mu}|S|-\vec{q},\epsilon_{d,\nu};\overrightarrow{k+p+q},\epsilon_{e,\mu}\rangle, \tag{1.151}$$

and,

$$-2(N^2-1)(\frac{d}{2}-1) =$$
$$b_1Z_3 + b_2Z_2' + b_3Z_{3g}\sum_p k^\mu\frac{1}{k^2p^2(p+k)^2}f^{abc}\langle\vec{k},\epsilon_{a,\nu};\vec{p},\epsilon_{b,\mu}|S|\overrightarrow{p+k},\epsilon_{c,\nu}\rangle +$$
$$b_4gZ_1'\sum k^\mu\frac{1}{p^2k^2(p+k)^2}f^{abc}\langle\vec{k},a;\vec{p},\epsilon_{b,\mu}|S|\overrightarrow{p+k},c\rangle. \tag{1.152}$$

First note that in the brackets of the Eqs. (1.151) and (1.152) appear the three point and four points vertex functions: second we did not introduce in the standard LSZ formulas the renormalization constant as we start with the renormalized Lagrangian and thus the corresponding propagators are those fixed by the renormalization conditions.

Eqs. (1.151) and (1.152) contain relations between the renormalization constants and the vertex functions. From these one can derive other useful relations. We shall analyze in detail what can one deduce from the first of them Eq. (1.151). First note that the propagators that appear in the LSZ reduction formula are on shell thus the right hand side of the Eq. (1.151) are divergent. Let us analyze a typical term,

$$a_2 g Z_{3g} g \sum_p k^\mu \frac{1}{k^2 p^2 (p+k)^2} f^{abc} \langle \vec{k}, \epsilon_{a,\nu}; \vec{p}, \epsilon_{b,\mu} | S | \overrightarrow{p+k}, \epsilon_{c,\nu} \rangle =$$
$$a_2 Z_{3g} g \sum_p k^2 \frac{1}{k^2 p^2 (p+k)^2} f^{abc} f^{abc} \Gamma(p, k, -(p+k)) + ... \tag{1.153}$$

Here $\Gamma(p, k, -(p+k))$ represents the vertex function from which we extracted the Lorentz and internal group dependence. Note that there are other terms on the right hand side of Eq. (1.153) which we ignore for reasons that will be evident soon. However since we work in the Feynman gauge and all momenta $k, p, p+k$ are on shell the corresponding vertex factor can depend only on p^2, k^2, pk which are zero so the corresponding vertex function can be assimilated with $\Gamma(0,0,0)$ which by the renormalization condition is simply $\Gamma(0,0,0) = g$. Note that the same argument would not apply for the case when the final states would contain fermions but even in this case when could just extract the convenient contribution from it. Then one can further write:

$$a_2 Z_{3g} \sum_p k^2 \frac{1}{k^2 p^2 (p+k)^2} f^{abc} f^{abc} \Gamma(p, k, -(p+k)) = a_2 Z_{3g} \sum_p \frac{1}{p^2 (p+k)^2} \text{const} g^2 =$$
$$a_2 Z_{3g} g^2 \text{const} \frac{(p^2)^2}{p^(k+p)^2} = b Z_{3g} g^2 \tag{1.154}$$

Applying the same procedure to all the terms in Eq. (1.151) one obtains:

$$1 = a Z_2 + b Z_{3g} g^2 + c Z_1'^2 g^2 + d Z_{4g} g^4, \tag{1.155}$$

where the coefficients a, b, c, d are independent of the gauge coupling constant but remain undetermined. The same arguments applied to Eq. (1.152) lead to:

$$x = y Z_3 + z Z_2 + u Z_{3g} g^2 + w Z_1' g^2 \tag{1.156}$$

To the relation obtained in Eqs. (1.155) and (1.156) one can add another one. This is based on writing $\frac{d\bar{c}^a(k)}{d\bar{c}^a(k)} = 1$ and using the property of integration of an anticommuting variable:

$$\int d\theta[A + B\theta] = B$$
$$\int \theta \frac{d}{d\theta}[A + B\theta] = B. \tag{1.157}$$

Applied to the partition function Z_0 and to the ghost fields Eq. (1.157) leads to another useful relation between the renormalization constants for the ghost fields:

$$r_1 = r_2 Z_2 + r_3 Z_1' g^2, \tag{1.158}$$

where r_1, r_2 and r_3 are constant independent of the gauge coupling constant.

In the standard approach the Eqs. (1.155), (1.156) and (1.158) are useful relations but are not enough for determining the beta function as we have three equations and five renormalization constants. However in the background gauge field method where the relations in Eq. (1.146) hold the number of renormalization constants is reduced to two and one can find important information. Thus one can extract directly the connection between the renormalization constants from the equations (1.155), (1.156) and (1.158) which will be rewritten as:

$$\begin{aligned} 1 &= (f_1 + f_2 g^2 + f_3 g^4) Z_3 + f_4 Z_2 g^2 \\ 1 &= (h_1 + h_2 g^2) Z_3 + h_4 Z_2 g^2 \\ 1 &= (c_1 + c_2 g^2) Z_2 \end{aligned} \tag{1.159}$$

where f_i, h_i and c_i are constants, some of them divergent. From the above system one determines a formula for Z_3 and a consistency condition:

$$\begin{aligned} Z_3 &= \frac{c_1 + (c_2 - h_4) g^2}{(c_1 + c_2 g^2)(h_1 + h_2 g^2)} \\ \frac{c_1 + (c_2 - h_4) g^2}{h_1 + h_2 g^2} &= \frac{c_1 + (c_2 - f_4) g^2}{f_1 + f_2 g^2 + f_3 g^4} \end{aligned} \tag{1.160}$$

The consistency condition should be regarded as an expansion in the gauge coupling constant and leads to relations among the coefficients from which one can extract the only one that simplifies Z_3 which is $c_2 = h_4$. Then Z_3 becomes:

$$Z_3 = \frac{1}{1 + d_1 g^2 + d_2 g^4}, \tag{1.161}$$

where $d_1 = h_2 + h_1 c_2/c_1$, $d_2 = c_2/c_1 h_2$ and we took $h_1 = 1$. The fact that $h_1 = f_1 = 1$ can be deduced from the initial equations and also from the known form of the renormalization constant.

We shall work in the dimensional regularization scheme where one can write the renormalization constant Z_3 as:

$$Z_1 = 1 + \frac{Z_3^{(1)}}{\epsilon} + \frac{Z_3^{(2)}}{\epsilon^2} + ... \tag{1.162}$$

Moreover the coefficients d_1 and d_2 are divergent and can be written as:

$$\begin{aligned} d_1 &= \frac{d_{(1)}}{\epsilon} + \frac{d_1^{(2)}}{\epsilon^2} + ... \\ d_2 &= \frac{d_2^{(2)}}{\epsilon} + \frac{d_2^{(2)}}{\epsilon^2} + ... \end{aligned} \tag{1.163}$$

Applying Eqs. (1.162) and (1.163) to Eq. (1.232) one obtains:

$$Z_1^{(1)} = -d_1^{(1)} g^2 - d_2^{(1)} g^4 \tag{1.164}$$

In dimensional regularization in the background gauge field method the beta function is defined as:

$$\beta = \mu^2 \frac{dg^2}{d\mu^2} = -g^4 \frac{\partial Z_3^{(1)}}{\partial g^2} = g^4 (d_1^{(1)} + g^2 d_2^{(2)}) \tag{1.165}$$

Since the two first order coefficients are universal the beta function is determined completely and thus corresponds to the 't Hooft scheme. The coefficients $d_1^{(1)}$ and $d_2^{(1)}$ are then identified with:

$$\begin{aligned} d_1^{(1)} &= -\frac{11}{3} N \frac{1}{(4\pi)^2} \\ d_2^{(1)} &= -\frac{34}{3} N^2 \frac{1}{(4\pi)^4}. \end{aligned} \tag{1.166}$$

Note that we determined an all order shape of the beta function without computing anything that resembles a Feynman diagram simply by using global properties of the partition function of a Yang Mills theory.

E. Higher orders or all orders beta function for nonabelian gauge theories with fermions.

The running of the gauge coupling constant with the scale is generally computed in quantum field theories using a perturbative approach which consists in expansion in a small parameter. In the dimensional regularization scheme beta function for QED is known at the fourth order whereas that for QCD at the fifth one. It is known that the first two order coefficients are renormalization scheme independent whereas the next ones depend on the specific renormalization procedure. There is also the 't Hooft scheme in which the beta function stops at two loops.

In a previous work we used a semi perturbative technique to compute the beta function for QED to obtain that the beta function stops at two loops. This method was further simplified and improved in a subsequent work where the exact form of the beta function for the Yang Mills theory has been determined from properties of the partition function and various correlators and by the use of the LSZ theorem. In the present work we extend our method to the more intricate case of the beta function of QCD with an arbitrary number of flavors. Whereas this case require more work the principles settled in our previous work remain unaltered. We determine that the all orders beta function stops at the two first orders coefficients. Note that this result is obtained without using any Feynman diagram or expansion in a small parameter.

We start with the gauge fixed Lagrangian for an $SU(N)$ gauge theory with N_f fermions in the fundamental representation:

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{c}^a(-\partial^\mu\partial_\mu - gf^{abc}\partial^\mu A^b_\mu)c^c + \sum_f \bar{\Psi}(i\gamma^\mu D_\mu - m_f)\Psi, \tag{1.167}$$

where,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu, \tag{1.168}$$

and,

$$D_\mu = \partial_\mu - igA^a_\mu t^a. \tag{1.169}$$

Here t^a is the generator of $SU(N)$ in the fundamental representation and for simplicity we shall consider $m_f = 0$. We shall work in the Feynman gauge ($\xi = 1$). One can express all the fields in the Fourier space:

$$\begin{aligned} A^a_\mu(x) &= \frac{1}{V}\sum_n \exp[-ik_n x]A^a_\mu(k_n) \\ \Psi(x) &= \frac{1}{V}\sum_m \exp[-ik_m x]\Psi(k_m) \\ c^b(x) &= \frac{1}{V}\sum_p \exp[-ik_p x]c^b(k_p). \end{aligned} \tag{1.170}$$

Then the Lagrangian takes the form:

$$\int d^4x\mathcal{L} = -\frac{1}{2}\frac{1}{V}\sum_n k_n^2 A^{a\nu}(k_n)A^a_\nu(-k_n) + \frac{1}{V}\sum_n k_n^2 \bar{c}^a(k_n)c^a(-k_n) +$$
$$+\frac{i}{V^2}g\sum_{n,m} k_n^\mu A^a_\nu(k_n)f^{abc}A^b_\mu(k_m)A^{c\nu}(-k_n-k_m) -$$
$$-\frac{1}{V^3}g^2 f^{abc}f^{ade}\sum_{n,m,p} A^{b\mu}(k_n)A^{c\nu}(k_m)A^d_\mu(k_p)A^e_\nu(-k_n-k_m-k_p) -$$
$$-\frac{i}{V^2}\sum_{n,m} k_n^\mu \bar{c}^a(k_n)gf^{abc}A^b_\mu(k_m)c^c(-k_n-k_m) +$$
$$\frac{1}{V}\sum_f\sum_n \bar{\Psi}_f(k_n)\gamma^\mu k_{\mu n}\Psi_f(k_n) + \frac{1}{V^2}g\sum_f\sum_{n,m}\bar{\Psi}_f(k_n)\gamma^\mu A^a_\mu(k_n-k_m)t^a\Psi_f(k_m) \quad (1.171)$$

The zero current partition function has the form:

$$Z_0 = \int\prod_i\prod_j\prod_m\prod_{fl}\prod_n\prod_p dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)\times$$
$$\exp[i\int d^4x\mathcal{L}], \quad (1.172)$$

where the exponent is considered in the Fourier space.

It is useful at this stage to settle some of the properties of Z_0. It is known that Z_0 apart from a factor in front is given by the exponential of the sum of all disconnected diagrams:

$$Z_0 = \text{factor}\times\exp[\sum_i V_i] \quad (1.173)$$

where V_i is a typical disconnected diagram. Since the calculation is done in the absence of external sources all V_i diagrams are closed and contain summations over momenta (that appear in propagators or vertices) and thus do not depend at all on any momenta. The factor in front is a product obtained from integrating the gaussian integrals corresponding to the kinetic terms. The final result has thus the expression:

$$Z_0 = \text{const}\prod_i(k_i^2)^{N^2-1}\prod_j(k_j^2)^{-d/2(N^2-1)}\prod_p[\det(\gamma^\mu p_\mu - m)]^{N_fN}\exp[\sum_i V_i] \quad (1.174)$$

where N is coming from the Yang Mills group $SU(N)$ and the first factor corresponds to the ghosts, the second to the gluon fields and the third to the fermion fields.

First we consider the partition function in Eq. (1.228) and introduce in the integrand the quantity $\frac{dA^a_\nu(k)}{dA^a_\nu(k)}$ to obtain:

$$Z_0 = \int \prod_{fl}\prod_i\prod_j\prod_n\prod_p\prod_m dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)\exp[i\int d^4x\mathcal{L}] =$$
$$= \int \prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)\frac{dA^a_\nu(k)}{dA^a_\nu(k)}\exp[i\int d^4x\mathcal{L}] =$$
$$= \int \prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)\frac{d}{dA^a_\nu(k)}[A^a_\nu(k)\exp[i\int d^4x\mathcal{L}]] -$$
$$- \int \prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)A^a_\nu(k) \times$$
$$\frac{d}{dA^a_\nu(k)}\exp[i\int d^4x\mathcal{L}], \tag{1.175}$$

where $\prod_{fl}$ contains separate products over flavors and colors.

The first term on the right side of the Eq. (1.175),

$$[\int \prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^d_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_f(k_n)d\Psi(k_p)A^a_\nu(k) \times$$
$$\exp[i\int d^4x\mathcal{L}]]_{A^a_\nu(k)=+\infty} -$$
$$[\int \prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^d_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_f(k_n)d\Psi(k_p)A^a_\nu(k) \times$$
$$\exp[i\int d^4x\mathcal{L}]]_{A^a_\nu(k)=-\infty}, \tag{1.176}$$

is zero since the ϵ term in the kinetic term will lead to an exponential that goes to zero. Here the product satisfy the constraint: $A^d_\mu(k_i) \neq A^a_\mu(k))$.

The second contribution leads to the result:

$$Z_0 = \int \prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)(-i)[-\frac{k^2}{V}A^{a\nu}(k)A^a_\nu(-k) +$$
$$\frac{3i}{V^2}gk^\mu\sum_p f^{abc}A^a_\nu(k)A^b_\mu(p)A^{c\nu}(-k-p) - \frac{i}{V^2}g\sum_p p^\nu\bar{c}^b(p)f^{bac}A^a_\nu(k)c^c(-p-k) -$$
$$-\frac{1}{V^3}g^2f^{bac}f^{bde}\sum_{p,q}A^a_\nu(k)A^c_\mu(p)A^{d\nu}(q)A^{e\mu}(-p-k-q) +$$
$$g\frac{1}{V^2}\sum_p\bar{\Psi}(p)\gamma^\mu t^a A^a_\mu(k)\Psi(-p-k)] \times \exp[i\int d^4x\mathcal{L}]. \tag{1.177}$$

We apply the same procedure to the partition function but this time introduce in the integrand the quantity $\frac{d\bar{\Psi}^r_{f_1}}{d\bar{\Psi}^r_{f_1}}$, where f is a flavor index and r is a color one. This yields:

$$Z_0 = -i\int\prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)\times$$
$$[\frac{1}{V}\bar{\Psi}^r_f\gamma^\mu k_\mu\Psi^r_f + \frac{1}{v^2}\sum_p\bar{\Psi}^r_f(k)\gamma^\mu p_\mu t^a_{rj}A^a_\mu(-p+k)\Psi^j_f(p)]\exp[i\int d^4x\mathcal{L}]. \tag{1.178}$$

Here we used the fact:

$$-i[\int\prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)\bar{\Psi}^r_{f_1}(k)\times$$
$$\exp[i\int d^4x\mathcal{L}]]_{\bar{\Psi}^i_f(k)\rightarrow\pm\infty} = 0 \tag{1.179}$$

since the spinors fields anticommute and there is no pairing for $\Psi^r_{f_1}(k)$ (there is no integration over $\bar{\Psi}^r_{f_1}(k)$) and thus the result is zero.

A similar procedure applied to the ghost field $c^c(k)$ leads to:

$$Z_0 = -i\int\prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)\times$$
$$[\frac{1}{V}k^2\bar{c}^c(k)k^2c^c(k) - \frac{i}{V^2}g\sum_p\bar{c}^a(p)p^\mu f^{abc}c^c(k)A^b_\mu(p-k)]\exp[i\int d^4x\mathcal{L}]. \tag{1.180}$$

Next we apply the operator $k^\mu\frac{d}{dk^\mu}$ to the Eq. (1.228) to obtain:

$$k^\mu\frac{dZ_0}{dk^\mu} = \int\prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)\times$$
$$i[-\frac{1}{V}k^2A^{a\nu}(k)A^a_\nu(-k) + \frac{2}{V}k^2\bar{c}^a(k)c^a(-k) + \frac{i}{V^2}k^\mu\sum_p A^a_\nu(k)f^{abc}gA^b_\mu(p)A^{c\nu}(-p-k) -$$
$$-\frac{i}{V^2}\sum_p k^\mu\bar{c}^a(k)gf^{abc}A^b_\mu(p)c^c(-p-k) + \frac{1}{V}\bar{\Psi}(k)\gamma^\mu k_\mu\Psi(k)]\times\exp[i\int d^4x\mathcal{L}], \tag{1.181}$$

where from Eq. (1.238) we calculate:

$$k^\mu\frac{dZ_0}{\partial k^\mu} = \left[N_fN - 2(N^2-1)[\frac{d}{2}-1]\right]Z_0. \tag{1.182}$$

Next we shall consider all the results from the perspective of renormalization. Thus the renormalized Lagrangian is:

$$\int d^4x\mathcal{L}_r = -\frac{1}{2}\frac{1}{V}Z_3\sum_n k_n^2 A^{a\nu}(k_n)A^a_\nu(-k_n) + \frac{1}{V}Z_1\sum_n k_n^2\bar{c}^a(k_n)c^a(-k_n) +$$
$$+\frac{i}{V^2}Z_{3g}g\sum_{n,m} k_n^\mu A^a_\nu(k_n)f^{abc}A^b_\mu(k_m)A^{c\nu}(-k_n-k_m) -$$
$$-\frac{1}{V^3}Z_{4g}g^2 f^{abc}f^{ade}\sum_{n,m,p} A^{b\mu}(k_n)A^{c\nu}(k_m)A^d_\mu(k_p)A^e_\nu(-k_n-k_m-k_p) -$$
$$-\frac{i}{V^2}Z_1'\sum_{n,m} k_n^\mu\bar{c}^a(k_n)gf^{abc}A^b_\mu(k_m)c^c(-k_n-k_m) +$$
$$\frac{1}{V}Z_2\sum_f\sum_n \bar{\Psi}_f(k_n)\gamma^\mu k_{\mu n}\Psi_f(k_n) + \frac{1}{V^2}gZ_2'\sum_f\sum_{n,m}\bar{\Psi}_f(k_n)\gamma^\mu A^a_\mu t^a\Psi(k_m)_f. \quad (1.183)$$

where for simplicity we drop the index r from the renormalized fields.

Then Eq. (1.177) will become:

$$Z_0 = \int\prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)(-i)[-Z_3\frac{k^2}{V}A^{a\nu}(k)A^a_\nu(-k) +$$
$$Z_{3g}\frac{3i}{V^2}gk^\mu\sum_p f^{abc}A^a_\nu(k)A^b_\mu(p)A^{c\nu}(-k-p) - \frac{i}{V^2}Z_1'g\sum_p p^\nu\bar{c}^b(p)f^{bac}A^a_\nu(k)c^c(-p-k) -$$
$$-\frac{1}{V^3}g^2Z_{4g}f^{bac}f^{bde}\sum_{p,q}A^a_\nu(k)A^c_\mu(p)A^{d\nu}(q)A^{e\mu}(-p-k-q) +$$
$$g\frac{1}{V^2}Z_2'g\sum_p\bar{\Psi}(p)\gamma^\mu t^a A^a_\mu(k)\Psi(-p-k)] \times \exp[i\int d^4x\mathcal{L}]. \quad (1.184)$$

Eq. (1.178) will transform to,

$$Z_0 = -i\int\prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p) \times$$
$$[\frac{1}{V}Z_2\bar{\Psi}^r_f\gamma^\mu k_\mu\Psi^r_f + \frac{1}{V^2}Z_2'g\sum_p\bar{\Psi}^r_f(k)\gamma^\mu p_\mu t^a_{rj}A^a_\mu(-p+k)\Psi^j_f(p)] \times$$
$$\exp[i\int d^4x\mathcal{L}]. \quad (1.185)$$

whereas Eq. (1.180) yields:

$$Z_0 = -i\int\prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA^a_\mu(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p) \times$$
$$[\frac{1}{V}Z_1k^2\bar{c}^c(k)k^2c^c(k) - \frac{i}{V^2}gZ_1'\sum_p\bar{c}^a(p)p^\mu f^{abc}c^c(k)A^b_\mu(p-k)] \times$$
$$\exp[i\int d^4x\mathcal{L}]. \quad (1.186)$$

Finally Eq. (1.181) will lead to:

$$k^{\mu}\frac{dZ_0}{dk^{\mu}}=\int\prod_{fl}\prod_{i}\prod_{j}\prod_{m}\prod_{n}\prod_{p}dA^a_{\mu}(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)\times$$
$$i[-\frac{1}{V}Z_3k^2A^{a\nu}(k)A^a_{\nu}(-k)+Z_1\frac{2}{V}k^2\bar{c}^a(k)c^a(-k)+\frac{i}{V^2}Z_{3g}k^{\mu}\sum_p A^a_{\nu}(k)f^{abc}gA^b_{\mu}(p)A^{c\nu}(-p-k)-$$
$$-\frac{i}{V^2}Z_1'\sum_p k^{\mu}\bar{c}^a(k)gf^{abc}A^b_{\mu}(p)c^c(-p-k)+\frac{1}{V}Z_2\bar{\Psi}(k)\gamma^{\mu}k_{\mu}\Psi(k)]\times\exp[i\int d^4x\mathcal{L}]. \qquad (1.187)$$

In the path integral formalism the two point gluon function has the expression:

$$\langle\Omega|T[A^a_{\mu}(x_1)A^b_{\nu}(x_2)]|\Omega\rangle=$$
$$\lim_{T\to\infty(1-i\epsilon)}\frac{\int dA^c_{\rho}d\bar{c}^d dc^e d\bar{\Psi}d\Psi A^a_{\mu}(x_1)A^b_{\nu}(x_2)\exp[i\int d^4x\mathcal{L}]}{\int dA^c_{\rho}d\bar{c}^d dc^e d\bar{\Psi}d\Psi\exp[i\int d^4x\mathcal{L}]}. \qquad (1.188)$$

We apply the LSZ reduction formula in the path integral formalism and in the Fourier space:

$$\langle\Omega|T[A^a_{\mu}(p_1)...A^d_{\nu}(p_m)A^b_{\rho}(k_1)...A^e_{\sigma}(k_n)]|\Omega\rangle\sim$$
$$\sim_{p_i^0(k_j^0)\to E_{\vec{p}_i}(E_{\vec{k}_j})}\text{polarization factor}\times\text{const}\times\langle\vec{p}_1...\vec{p}_m|S|\vec{k}_1...\vec{k}_n\rangle\times$$
$$\left(\prod_{i=1}^m\frac{iZ_3^{1/2}}{p_i^2+i\epsilon}\right)\left(\prod_{j=1}^n\frac{iZ_3^{1/2}}{k_j^2+i\epsilon}\right). \qquad (1.189)$$

In a previous work we illustrated in detail how we apply this formula to the gauge and ghost terms in the relations in Eqs. (1.184), (1.185), (1.186) and (1.187).

LSZ formula is more intricate and complicated for fermions as it can be seen from the following equation for a process with two initial and two final fermions:

$$_{out}\langle f|i\rangle_{in}=\langle f|S|i\rangle\approx\int d^4x_1d^4x_2d^4y_1d^4y_2\exp[-ik_1y_1][\bar{u}_{s_1'}(k_1)(-i\gamma^{\mu}\partial_{\mu y_1}+m)]_{\beta_1}\times$$
$$\exp[-ik_2y_2][\bar{u}_{s_2'}(k_2)(-i\gamma^{\mu}\partial_{\mu y_2}+m)]_{\beta_2}\times$$
$$\langle 0|T\Psi_{\beta_2}(y_2)\Psi_{\beta_1}(y_1)\bar{\Psi}_{\alpha_1}(x_1)\bar{\Psi}_{\alpha_2}(x_2)|0\rangle\times$$
$$[(i\gamma^{\mu}\overleftarrow{\partial}_{\mu x_1}+m)u_{s_1}(p_1)]_{\alpha_1}]\exp[ip_1x_1]\times$$
$$[(i\gamma^{\mu}\overleftarrow{\partial}_{\mu x_2}+m)u_{s_2}(p_2)]_{\alpha_2}]\exp[ip_2x_2]. \qquad (1.190)$$

Here α_1, α_2, β_1 and β_2 are spinor indices and all momenta are on shell.

This formula is too intricate to be easily applicable to our calculations. If $a^{\dagger}_s(\vec{p}$ and $b^{\dagger}_s(\vec{p})$

are the operators that create a one particle state with charge 1 respectively -1 one can write:

$$
\begin{aligned}
a_s^\dagger(\vec{p})_{in} &\to i\int d^4x\bar{\Psi}(x)(i\gamma^\mu\overleftarrow{\partial}_\mu + m)u_s(\vec{p})\exp[ipx] \\
a_s(\vec{p})_{out} &\to i\int d^4x\exp[-ipx]\bar{u}_s(\vec{p}))(-i\gamma^\mu\partial_\mu + m)\Psi(x) \\
b_s^\dagger(\vec{p})_{in} &\to i\int d^4x\exp[ipx]\bar{v}_s(\vec{p}))(-i\gamma^\mu\partial_\mu + m)\Psi(x) \\
b_s(\vec{p})_{out} &\to i\int d^4x\bar{\Psi}(x)(i\gamma^\mu\overleftarrow{\partial}_\mu + m)v_s(\vec{p})\exp[-ipx].
\end{aligned}
\tag{1.191}
$$

Let us rewrite the first equation in (1.191) in the Fourier:

$$
\begin{aligned}
&a_s^\dagger(\vec{p}) = i\int d^4x\int\frac{d^4k}{(2\pi)^4}\bar{\Psi}(k)\exp[ikx](i\gamma^\mu\overleftarrow{\partial}_\mu + m)u_s(\vec{p})\exp[ipx] = \\
&i\int d^4x\int\frac{d^4k}{(2\pi)^4}\exp[ikx]\bar{\Psi}(k)(-\gamma^\mu k_\mu + m)u_s(\vec{p})\exp[ipx] = \\
&i\bar{\Psi}(p)(\gamma^\mu p_\mu + m)u_s(\vec{p}),
\end{aligned}
\tag{1.192}
$$

The above formula is still useless as we need to express $\bar{\Psi}(\vec{p})$ in terms of the other quantities. In order to solve that we consider the sum:

$$
\sum_s a_s^\dagger(\vec{p})_{in}\bar{u}_s(\vec{p}) = \sum_s i\bar{\Psi}(p)(\gamma^\mu p_\mu + m)u_s(\vec{p})\bar{u}_s(\vec{p}). \tag{1.193}
$$

Knowing that the following formula holds,

$$
\sum_s u_s(\vec{p})\bar{u}_s(\vec{p}) = (-\gamma^\mu p_\mu + m), \tag{1.194}
$$

we obtain:

$$
\sum_s a_s^\dagger(\vec{p})_{in}\bar{u}_s(\vec{p}) = -i\bar{\Psi}(p)(p^2 - m^2). \tag{1.195}
$$

From Eq. (1.195) we extract:

$$
\bar{\Psi}(p) = i\sum_s a_s^\dagger(\vec{p})_{in}\bar{u}_s(\vec{p})\frac{1}{p^2 - m^2}. \tag{1.196}
$$

We shall take $m_f = 0$ in all subsequent calculations.

We shall consider only the fermion fields. First we divide Eq. (1.184) by Z_0 which yields:

$$
\begin{aligned}
&1 = \text{terms that do not involve fermions} + \\
&\frac{1}{Z_0}\int\prod_{fl}\prod_i\prod_j\prod_m\prod_n\prod_p dA_\mu^a(k_i)d\bar{c}^b(k_j)dc^d(k_m)d\bar{\Psi}_{fl}(k_n)d\Psi_{fl}(k_p)\times \\
&(-i)g\frac{1}{V^2}Z_2'\sum_p\bar{\Psi}(p)\gamma^\mu t_r^a A_\mu^a\Psi(p-k)\exp[i\int d^4x\mathcal{L}].
\end{aligned}
\tag{1.197}
$$

But the last term in Eq.(1.197) is just :

$$(-i)g\frac{1}{V^2}Z_2'\sum_p\langle\Omega|T[\bar{\Psi}(p)\gamma^\mu t_r^a A_\mu^a(k)\Psi(p-k)|\Omega\rangle. \tag{1.198}$$

Then by applying Eqs. (1.190) and (1.196) to Eq. (1.198) one obtains for Eq. (1.197):

$$1 = \text{terms that do not involve fermions} + (-i)g\frac{1}{V^2}Z_2'\sum_p\frac{1}{p^2(p+k)^2k^2}\times\text{const}$$
$$\sum_{s,s'}(t_r^a)_{ij}\bar{u}_s(\vec{p})\langle(\vec{p},s)_i;\vec{k}\epsilon^a_{k\mu}|\gamma^\mu S|((p+k),s')_j\rangle u_{s'}(\overrightarrow{p+k}). \tag{1.199}$$

Note that in the above equation the term in brackets actually contains the vertex function V_{ij}^a which is known by the renormalization conditions. Then one can write:

$$\bar{u}(\vec{p})V_{ji}^{a\mu}\gamma_\mu(t_r^a)_{ij}u_s(p\vec{+}k)\approx\frac{N^2-1}{2}gp^2 \tag{1.200}$$

where we used the fact that for an on shell fermion $p^2 = m^2$ and also as defined in the present work the vertex function ($V_{ji}^{a\mu} = gt_{ji}^a p^\mu$)contains already a compression between two fermion states. Also note that the factor $\sum_p\frac{p^2}{k^2p^2(k+p)^2}$ in the limit of on shell states leads to a constant.

Since a similar procedure applies to all the fields and interaction that appear in Eqs. (1.184), (1.185), (1.186) and (1.187) these relations will become:

$$\begin{aligned}
1 &= aZ_3 + bZ_{3g}g^2 + cZ_1'g^2 + dZ_{4g}g^4 + gZ_2'g^2\\
s_1 &= s_2Z_2 + s_3Z_2'g^2\\
r_1 &= r_2Z_1 + r_3Z_1'g^2\\
x &= yZ_3 + zZ_1 + qZ_2 + uZ_{3g}g^2 + wZ_1'g^2,
\end{aligned} \tag{1.201}$$

where we absorbed all the constants in front of the terms in the new coefficients a, b, c, d, e, x, y, z, q, u, v, w, r_1, r_2, r_3, s_1, s_2, s_3.

In general in the dimensional regularization scheme similar relations exist also in other schemes) the renormalization constants satisfy the Slanov Taylor identities:

$$g_0^2 = \frac{Z_{3g}^2}{Z_3^3}g^2\mu^\epsilon = \frac{Z_{4g}}{Z_3^2}g^2\mu^\epsilon = \frac{Z_1'^2}{Z_1^2Z_3}g^2\mu^\epsilon = \frac{Z_2'^2}{Z_2^2Z_3}g^2\mu^\epsilon, \tag{1.202}$$

where $d = 4-\epsilon$ and μ is a parameter with dimension of mass. In the background gauge field method there is a great simplification given by the relations:

$$
\begin{aligned}
Z_1 &= Z_1' \\
Z_2 &= Z_2' \\
Z_3 &= Z_{3g} = Z_{4g}.
\end{aligned}
\tag{1.203}
$$

Then one can write the four relations in Eq. (1.201) in a more compact form:

$$
\begin{aligned}
1 &= (f_1 + f_2 g^2 + f_3 g^4) Z_3 + f_4 Z_1 g^2 + f_5 Z_2 g^2 \\
1 &= (t_1 + t_2 g^2) Z_2 \\
1 &= (c_1 + c_2 g^2) Z_1 \\
1 &= (h_1 + h_2 g^2) Z_3 + Z_1 (h_3 + h_4 g^2) + h_5 Z_2
\end{aligned}
\tag{1.204}
$$

From the last three equations in Eq. (1.204) we determine:

$$
Z_3 = \frac{(c_1 t_1 - h_3 t_1 - h_5 c_1) + (c_2 t_1 + c_1 t_2 - h_3 t_2 - h_4 t_1 - h_5 c_2) g^2 + (c_2 t_2 - h_4 t_2) g^4}{(h_1 + h_2 g^2)(c_1 + c_2 g^2)(t_1 + t_2 g^2)} \tag{1.205}
$$

From the first three equations in Eq. (1.204) we compute:

$$
Z_3 = \frac{c_1 t_1 + (c_2 t_1 + c_1 t_2 - f_4 t_1 - f_5 c_1) g^2 + (c_2 t_2 - f_4 t_2 - f_5 c_2) g^4}{(f_1 + f_2 g^2 + f_3 g^4)(c_1 + c_2 g^2)(t_1 + t_2 g^2)}. \tag{1.206}
$$

We shall use this last equation as a constraint. Matching the order of the coefficients with those in Eq. (1.205) we obtain the condition:

$$
(c_2 t_2 - h_4 t_2) f_3 = 0 \tag{1.207}
$$

from which we deduce $c_2 = h_4$ since none of the coefficients are allowed to be zero.

There are some coefficients in Eq. (1.204) that can be determined directly form the preceding defining equations. These are those associated with the terms involving the gluon, fermion or ghost two point functions. Thus from Eq. (1.187) and the subsequent versions of it (noting that h_1, h_3 and h_5 are associated with the two point function for gluon, ghost and fermion respectively) one can compute by simple gaussian integration:

$$
\begin{aligned}
h_1 &= \frac{-4(N^2-1)}{NN_f - 2(N^2-1)} \\
h_3 &= \frac{2(N^2-1)}{NN_f - 2(N^2-1)} \\
h_5 &= \frac{N_f N}{NN_f - 2(N^2-1)}.
\end{aligned}
\tag{1.208}
$$

Furthermore Eq. (1.208) leads to the following useful recurrence relation:

$$1 - h_3 - h_5 = h_1. \tag{1.209}$$

Similarly from Eqs. (1.178) and (1.186) one can determine $c_1 = t_1 = 1$ for the same reasons. By substituting in Eq.(1.205) the correct values for h_1, h_3, h_5, c_1, t_1 the expression for Z_3 becomes:

$$Z_3 = \frac{1 + u_1 g^2}{1 + v_1 g^2 + v_2 g^4 + v_3 g^6}, \tag{1.210}$$

where,

$$\begin{aligned}
u_1 &= \frac{t_2}{h_1} - \frac{h_3 t_2}{h_1} - \frac{h_5 c_2}{h_1} \\
v_1 &= c_2 + t_2 + \frac{h_2}{h_1} \\
v_2 &= \frac{h_2 t_2}{h_1} + c_2 t_2 + \frac{h_2 c_2}{h_1} \\
v_3 &= \frac{h_2 c_2 t_2}{h_1}.
\end{aligned} \tag{1.211}$$

In general in the dimensional regularization scheme the renormalization constant Z_3 can be written as:

$$Z_3 = 1 + \sum_{n=1}^{\infty} \frac{Z_3^{(n)}}{\epsilon^n}. \tag{1.212}$$

Note that similar expression exist for any renormalization scheme by making simple substitutions for ϵ.

The beta function is defined as:

$$\beta(g^2) = \mu^2 \frac{dg^2}{d\mu^2} = -\frac{1}{2} g^3 \frac{\partial Z_3^{(1)}}{\partial g} = -g^4 \frac{\partial Z_3^{(1)}}{\partial g^2}. \tag{1.213}$$

We identify Eq. (1.210) with Eq. (1.212) noting that the degree of divergence (given by powers in $\frac{1}{\epsilon}$) of the coefficients u_1, v_1, v_2, v_3 is zero, one or greater than one:

$$\begin{aligned}
&[1 + \frac{Z_3^{(1)}}{\epsilon} + \frac{Z_3^{(2)}}{\epsilon^2} + ...][1 + v_1^{(0)} g^2 + \frac{v_1^{(1)}}{\epsilon} g^2 + v_2^{(0)} g^4 + \frac{v_2^{(1)}}{\epsilon} g^4 + v_3^{(0)} g^6 + \frac{v_3^{(1)}}{\epsilon} g^6 + ...] = \\
&u_1^{(0)} g^2 + \frac{u_1^{(1)}}{\epsilon} g^2 + ...
\end{aligned} \tag{1.214}$$

To order $\frac{1}{\epsilon}$ this leads to:

$$\frac{Z_3^{(1)}}{\epsilon}[1 + v_1^{(0)} g^2] + \frac{v_1^{(1)}}{\epsilon} g^2 + \frac{v_2^{(1)}}{\epsilon} g^4 + \frac{v_3^{(1)}}{\epsilon} g^6 = \frac{u_1^{(1)}}{\epsilon} g^2 \tag{1.215}$$

Here,

$$v_1^{(0)} = c_2^{(0)} + t_2^{(0)} + \frac{h_2^{(0)}}{h_1}$$
$$v_3^{(1)} = \frac{(h_2 c_2 t_2)^{(0)}}{h_1}. \tag{1.216}$$

Using Eq. (1.204) and the constraint $c_2 = h_4$ by considering simple expansion in powers of $\frac{1}{\epsilon}$ one obtains that $c_2^{(0)} = t_2^{(0)} = h_2^{(0)} = 0$. Eq. (1.216) leads also to $v_3^{(1)} = 0$. Then from Eq. (1.215) the dependence of $Z_3^{(1)}$ on the coupling constant emerges:

$$Z_3^{(1)} = \beta_0 g^2 + \beta_1 g^4, \tag{1.217}$$

where β_0 and β_1 are coefficients independent of g^2. We thus determined the all order shape of the beta function of QCD only by using global properties of the partition function and of the various two, three or four point correlators.

According to Eq. (1.213) the beta function contains only the first two order renormalization scheme independent coefficients and is given by:

$$\beta(g^2) = \frac{dg^2}{d\ln\mu^2} =$$
$$-[\frac{11}{3}N - \frac{2}{3}N_f]\frac{g^4}{16\pi^2} - [\frac{34}{3}N^2 - 2\frac{N^2-1}{2N}N_f - \frac{10}{3}NN_f]\frac{g^6}{256\pi^4}. \tag{1.218}$$

F. A new perspective on the phase transitions for gauge theories.

The $U(1)$ abelian Higgs model is one of the simplest theories that contains both gauge fields and matter. The phase structure for this model has been studied at zero temperature both for compact Higgs $|H| = \text{const}$ or for varying $|H|$. The finite temperature regime has also been analyzed. It was shown in all these instances that the model displays three main phases: a Coulomb phase where the potential is $V(r) \approx \frac{1}{r}$; a Higgs phase where $V(r) \approx \text{const}$ and a confinement phase where $V(r) \approx r$. However for the particular case of a Higgs in the fundamental representation with charge unit it seems that there is no real distinction between the Higgs phase and the confinement one and thus the two of them are connected. All the above findings have been confirmed numerically through Monte Carlo simulations.

In the present work we shall revisit the abelian Higgs model without making any constraining assumptions by considering its dual description in terms of fermion variables. We suggest that this alternate description may alter the structure of the Lagrangian expressed in terms of the original variables. The relevant parameters are the coefficients of the terms in the modified Lagrangian and their relative magnitudes reveals the exact phase in which the system is in. We then use the standard path integral approach to show that the Lorentz invariant function $\langle A^\mu(x)A_\mu(y)\rangle$ (but our arguments would work as well for the regular two point function) is a reasonable order parameter to indicate the behavior of the model in different phases. Although we use a novel perspective our findings agree well with the standard knowledge in the field.

We start with the $U(1)$ abelian Higgs model given by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D^\mu\Phi^* D_\mu\Phi - V(\Phi), \tag{1.219}$$

where,

$$\begin{aligned} D_\mu\Phi &= (\partial_\mu + ieA_\mu)\Phi \\ V(\Phi) &= m^2\Phi^*\Phi + \frac{\lambda}{2}(\Phi^*\Phi)^2. \end{aligned} \tag{1.220}$$

Upon expansion of the gauge kinetic term for the scalar field the Lagrangian in Eq. (1.219) becomes:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial^\mu\Phi^*\partial_\mu\Phi - ieA^\mu\Phi^*\partial_\mu\Phi + ieA_\mu\Phi\partial^\mu\Phi^* + \mathcal{L}_1 \\ \mathcal{L}_1 &= e^2 A_\mu A^\mu\Phi^*\Phi - m^2\Phi^*\Phi - \frac{\lambda}{2}(\Phi^*\Phi)^2. \end{aligned} \tag{1.221}$$

Now consider the dual description of the scalar and gauge fields in terms of fermion variables:

$$\begin{aligned} A_\mu &= \frac{1}{M^2}\bar{\Psi}\gamma_\mu\Psi \\ \Phi &= \frac{1}{M^2}(\bar{\Psi}\Psi + \bar{\Psi}\gamma^5\Psi) \end{aligned} \tag{1.222}$$

Note that we can do this and still preserve the gauge invariance. Moreover the corresponding bilinear forms can be considered independent since an on-shell fermions has four degrees of freedom, an on shell massless gauge boson has two and the complex scalar has two. This means that the redefinition of the fields made in Eq. (1.222) matches the number of original degrees of freedom for the Lagrangian in Eq. (1.219).

We shall work with the Lagrangian $\mathcal{L}$ in which we ignore the λ interaction term and the scalar mass m^2 is an infinitesimal parameter. We consider the Fierz transformation,

$$\Phi\Phi^* = \frac{1}{M^2}\left[\bar{\Psi}\Psi\bar{\Psi}\Psi - \bar{\Psi}\gamma^5\Psi\bar{\Psi}\gamma^5\Psi\right] =$$
$$= \frac{1}{2M^2}\left[\bar{\Psi}\gamma^\mu\Psi\bar{\Psi}\gamma_\mu\Psi - \bar{\Psi}\gamma^\mu\gamma^5\Psi\bar{\Psi}\gamma_\mu\gamma^5\Psi\right] = \frac{1}{2}[A^\mu A_\mu - A^{\mu 5}A^5_\mu] \tag{1.223}$$

to write:

$$A^2_\mu = xA^2_\mu + (1-x)\left[2|\Phi|^2 + (A^5_\mu)^2\right]$$
$$|\Phi|^2 = y|\Phi|^2 + \frac{1-y}{2}[A^2_\mu - (A^5_\mu)^2], \tag{1.224}$$

where x and y are real parameters with $0 \le x \le 1$ and $0 \le y \le 1$. Then the Lagrangian $\mathcal{L}_1$ will become:

$$\mathcal{L}_1 = e^2\Big[xA^2_\mu + (1-x)[2|\Phi|^2 + (A^5_\mu)^2]\Big]\Big[y|\Phi|^2 + (1-y)/2[A^2_\mu - (A^5_\mu)^2]\Big] - m^2|\Phi|^2 =$$
$$= 2e^2y(1-x)|\Phi|^4 + e^2\frac{x(1-y)}{2}A_\mu A^\mu A_\nu A^\nu +$$
$$e^2\Big[xy + (1-x)(1-y)\Big]A^\mu A_\mu|\Phi|^2 - m^2|\Phi|^2. \tag{1.225}$$

Since the parameter m^2 is considered infinitesimal there is no need to make the replacement in Eq. (1.224) also for the term $m^2|\Phi|^2$. We solve the equation of motion for the auxiliary field A^5_μ to get $A^5_\mu = 0$. Thus in the last line of the equation we took $A^5_\mu = 0$. Net we shall analyze the phases of the Lagrangian described in Eq. (1.225) in term of the parameters x and y.

It is then clear that for the Lagrangian given in Eq. (1.225) the terms of interest are the $(A_\mu A^\mu)^2$ and $|\Phi|^4$ which after the rescaling of the fields with the constants in front of them will multiply the kinetic term for the scalar field with $1/[e^2y(1-x)]^{1/2}$ whereas that of the gauge field with $1/[e^2x(1-y)]^{1/2}$. Without loss of generality we shall consider the initial value of e^2 large such that by tuning x and y one can get any value of the effective coupling small or large.

Let us briefly explain our procedure. We denote $z_1 = [x(1-y)]^{1/4}$ and $z_2 = [y(1-x)]^{1/4}$. We rescale the fields as $z_1A_\mu = A'_\mu$ and $z_2\Phi = \Phi'$ and write the Lagrangian in terms of the new variable in order to put in evidence the relative magnitude of the kinetic terms. We shall use an unusual order parameter the Lorentz invariant two point function $\langle A^\mu(x)A_\mu(y)\rangle$ and work in the Feynman gauge. It is evident then that the order parameter is not gauge invariant. However we will show that the behavior of the model is very well indicated by this order parameter by using the standard functional approach. In order to compute the order parameter we will go back to the initial approximate Lagrangian where the variables $\Phi(x)$ and $A_\mu(x)$ are retrieved.

In conclusion we find four limiting cases:

1)**Higgs phase**

This phase is obtained for the following values of the parameter x and y: $x \approx 0$ and $y \approx 0$ or $x \approx 1$ and $y \approx 1$. In both these cases the full Lagrangian has the expression:

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4}\frac{1}{z_1^2}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{z_2^2}\partial^\mu\Phi'^*\partial_\mu\Phi' - \\
& -i\frac{e}{z_1 z_2^2}A'^\mu\Phi'^*\partial_\mu\Phi' + i\frac{e}{z_1 z_2^2}A'_\mu\Phi'\partial^\mu\Phi'^* + e^2 A'_\mu A'^\mu_1\Phi'^*\Phi' - \frac{m^2}{z_2^2}\Phi'^*\Phi' \qquad (1.226)
\end{aligned}$$

where $z_1 \approx 0$ and $z_2 \approx 0$. If we solve for the Higgs expectation value in the initial Lagrangian in Eq. (1.225) we find:

$$\langle\Phi^2\rangle = \frac{m^2}{4e^2 y(1-x)} = \text{large} \qquad (1.227)$$

Here it is considered that the limits of the parameters x and y supersede the limits small or large of the parameters m^2 and e^2. We this expect that this phase will correspond to the Higgs phase. Note that the vev of the scalar in Eq. (1.227) Introduced in Eq. (1.226) leads to a mass term for the gauge boson.

In order to show that indeed this situation corresponds to the Higgs phase we first rescale the Higgs field as $e\Phi \Rightarrow \Phi$. Then all the terms that contain the Higgs field in the Lagrangian in Eq. (1.226) will be further suppressed and can be neglected. Moreover we shall use the Feynman gauge and calculate the two point function for the corresponding Lagrangian.

First we need to compute the partition function in the Fourier space:

$$\begin{aligned}
Z \approx & \int dA_\mu(p)d\Phi(q)\exp\Bigg[-i\frac{1}{2}\sum_p A^\mu(p)p^2A_\mu(-p) + \\
& i\sum_{p,q,r}A^\mu(p)A_\mu(q)\Big[\Phi_1(r)\Phi(-p-r-q)_1 + \Phi(r)_2\Phi_2(-p-r-q)\Big]\Bigg], \qquad (1.228)
\end{aligned}$$

where $\Phi_1(x) = \mathrm{Re}\Phi(x)$ and $\Phi_2(x) = \mathrm{Im}\Phi(x)$. In order to find the two point function we shall use a trick. We change the variable $\Psi_1(p) = u\Phi_1(p)$ for any $p \neq q$ and $\Psi_2(p) = u\Phi_2(p)$

where q is arbitrary and fixed. The adimensional parameter u is considered very large and arbitrary. Upon neglecting the infinitesimal terms the partition function will become:

$$Z = \text{const}\frac{1}{u^N}\int dA_{\mu}(p)d\Phi_1(q)\exp\left[-i\frac{1}{2}\sum_p A^{\mu}(p)p^2A_{\mu}(p) + i\sum_p A^{\mu}(p)A_{\mu}(-p)\Phi_1(q)\Phi_1(-q)\right] =$$
$$= \text{const}\frac{1}{u^N}\int dA_{\mu}(r)\frac{1}{\sum_p A^{\mu}(p)A_{\mu}(-p)}\exp\left[-i\frac{1}{2}\sum_p A^{\mu}(p)p^2A_{\mu}(p)\right] \tag{1.229}$$

where we took into account that $\Phi(-p) = \Phi(p)^*$. Here N is a very large integer number related to the counting of the momenta on the lattice; for example all the momenta can be written as k_n^{μ} with $1 \leq n \leq N_{max}$ case in which $N = 2N_{max} - 1$ since we subtract $\Phi_1(q)$. Then using,

$$\sum_{p^2}\frac{\delta}{\delta p^2}Z = \text{const}\frac{1}{u^N}\int dA_{\mu}(p)\exp\left[-i\frac{1}{2}\sum_p A^{\mu}(p)p^2A_{\mu}(p)\right] = \text{const}\prod_p\frac{1}{(p^2)^2} \tag{1.230}$$

we obtain:

$$Z = a\prod_p\frac{1}{(p^2)^2} + b \tag{1.231}$$

where a and b are two constants independent on the momenta (For example b takes into account the fact that $Z \neq 0$ even for $p^2 = \infty$). The Lorentz invariant two point function in the Fourier space can then be written as (the momentum delta function can be included from the beginning in the equation if we take into account the initial expression of the partition function in Eq. (1.229)):

$$\langle A^{\mu}(p)A_{\mu}(-p)\rangle = \frac{1}{Z}\frac{\delta Z}{\delta p^2} = -2a\frac{1}{(p^2)^3}\frac{1}{a\prod_q\frac{1}{(q^2)^2} + b}\prod_{q\neq p}\frac{1}{(q^2)^2} \approx \text{const}\frac{1}{(p^2)^3}. \tag{1.232}$$

We need to justify the result in Eq. (1.232). For that we observe that whereas a is finite the quantity b measures the degree of divergence of the partition function so it is very large. Thus the a term can be neglected compared to the b one.

We are mainly interested to find the potential corresponding to the order parameter in the coordinate space between two sources. In this approach we need to consider as sources two scalar fields with the momenta q_1, q_2 such that $p^2 = (q_1 - q_2)^2 \approx |\vec{q_1} - \vec{q_2}|^2 = |\vec{p}|^2$. Then we can write directly:

$$V = \text{const}\frac{1}{b}\int_{-\infty}^{\infty} dq\frac{\exp[iqr]}{r}q\frac{1}{(q^2+\mu^2)^3} = \frac{\text{const}}{r}\exp\left[-\mu r\right]\left[r + O(\mu)\right] \approx \text{const} \tag{1.233}$$

Here the integral is done on a contour closed above in the complex plane with the calculation of the residue of the third order pole $q = i\mu$. The constant in front contains a product of factors that goes to infinity or zero that lead overall to a finite constant. If the limit μ is taken to zero one regains the standard result for the Higgs phase which says that the potential is constant.

In the end it is important to mention that the gauge field acquires a mass in the Higgs phase as usual although this is not manifest in our approach because we integrated over the shifted scalar field which corresponds to the initial fields in the Lagrangian and not to that resulting from the spontaneous symmetry breaking. Note also that in this approach the Higgs phase is present even in the absence of the λ term in the Lagrangian.

2)**Coulomb phase**

This phase is obtained for $y \approx 1$ and $x \approx 0$. The approximate Lagrangian is:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\frac{1}{z_1^2}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{z_2^2}\partial^\mu\Phi'^*\partial_\mu\Phi' - \\ &-i\frac{e}{z_1 z_2^2}A'^\mu\Phi'^*\partial_\mu\Phi' + i\frac{e}{z_1 z_2^2}eA'_\mu\Phi'\partial^\mu\Phi'^*,\end{aligned} \tag{1.234}$$

where $z_1 \approx 0$ and $z_2 =\approx 1$. There is no mass for the gauge boson and the vev of the Higgs is zero. With the change of variable $e\Phi \to \Phi$ the scalar field decouples and the final Lagrangian contains in first order only the kinetic term for the gauge field. The propagator is simply $\frac{1}{p^2}$ and the potential in the coordinate space is $V(r) \approx \frac{1}{r}$. This is the Coulomb phase of the abelian Higgs model.

3) **Higgs +confinement phase**

This case corresponds to $y \approx 0$ and $x \approx 1$ and has the Lagrangian:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\frac{1}{z_1^2}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{z_2^2}\partial^\mu\Phi'^*\partial_\mu\Phi' - \\ &-i\frac{e}{z_1 z_2^2}A'^\mu\Phi'^*\partial_\mu\Phi' + i\frac{e}{z_1 z_2^2}eA'_\mu\Phi'\partial^\mu\Phi'^* + e^2(A'^\mu A'_\mu)^2 - \frac{m^2}{z_2^2}\Phi'^*\Phi',\end{aligned} \tag{1.235}$$

where $z_1 \approx 1$ and $z_2 \approx 0$. The kinetic term for the Higgs scalar is very big whereas that for the gauge field is very small. Alternatively the interaction term $A_\mu A^\mu A_\nu A^\nu$ is very large. The Higgs expectation value is very big as it can be seen from Eq. (1.227). The structure of the Lagrangian that contains a large gauge quadrilinear term is specific to confinement (see the next subsection) and the large vev of the Higgs indicate that the system is in a combined Higgs confinement phase.

4)**Confinement phase**

The parameters x and y take the values: $y \approx 1/2$ and $y \approx 1/2$. The Lagrangian for this case has the expression:

$$\mathcal{L} = -\frac{1}{4}\frac{1}{z_1^2}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{z_2^2}\partial^{\mu}\Phi'^{*}\partial_{\mu}\Phi' -$$
$$-i\frac{e}{z_1 z_2^2}A'^{\mu}\Phi'^{*}\partial_{\mu}\Phi' + i\frac{e}{z_1 z_2^2}eA'_{\mu}\Phi'\partial^{\mu}\Phi'^{*} + \frac{1}{2}e^2A'_{\mu}A'^{\mu}\Phi'^{*}\Phi' - \frac{m^2}{z_2^2}\Phi'^{*}\Phi' + +e^2(A'^{\mu}A'_{\mu})^2 \quad (1.236)$$

where $z_1 \approx \sqrt{1/2}$ and $z_2 \approx \sqrt{1/2}$. In this case both the kinetic scalar term and the kinetic gauge terms are small. We first rescale the scalar field as $e\Phi \to \Phi$. Thus one can neglect all the terms containing the scalar field except that containing the quadrilinear interaction with the gauge field. The approximate Lagrangian in the old variables A_{μ} and Φ is:

$$\mathcal{L} = \frac{1}{2}A^{\nu}\partial^2 A_{\nu} + \frac{e^2}{8}(A^{\mu}A_{\mu})^2 + \frac{1}{2}A^{\mu}A_{\mu}|\Phi|^2, \quad (1.237)$$

where we considered the Feynman gauge. We shall use the same a approach as in Eqs. (1.229), (1.230) and (1.231) to deal with the $A^{\mu}A_{\mu}|\Phi|^2$ term and with the integral over $\Phi(x)$ to obtain:

$$Z = c + \int dA^{\mu}(x)\exp\left[i\int d^4x\frac{1}{2}A^{\nu}(x)\partial^2 A_{\nu}(x) + i\int d^4x\frac{e^2}{8}(A^{\mu}(x)A_{\mu}(x))^2\right], \quad (1.238)$$

where c is a large constant. We shall rewrite the second term on the right hand side of the Eq.(1.238) as:

$$\int dA^{\mu}(x) \times \exp\Bigg[i\Big[\int d^4x d^4y\frac{1}{2}A^{\nu}(x)\partial^2(x)\delta(x-y)A_{\nu}(y) +$$
$$\frac{e^2}{8}\int d^4x\int d^4y(A^{\mu}(x)A_{\mu}(y))^2\delta(x-y)\Big]\Bigg], \quad (1.239)$$

and define the operator $K(x,y) = \partial^2\delta(x-y)$. In order to determine the expression in Eq. (1.239) it is easier to work in the coordinate space. We write:

$$\int d^4x\int d^4y\frac{1}{2}A^{\nu}(x)\partial^2(x)\delta(x-y)A_{\nu}(y) + \frac{e^2}{8}\int d^4x\int d^4y(A^{\mu}(x)A_{\mu}(y))^2\delta(x-y) =$$
$$\int d^4x\int d^4y\Bigg[\frac{e^2}{8}\Big[A_{\mu}(x)A^{\mu}(y) + \frac{2}{e^2}\partial(y)\partial(x)\Big]^2\delta(x-y) -$$
$$\Big[\frac{1}{2e^2}\partial^2(x)\partial^2(y)\delta(x-y)\Big]\Bigg]. \quad (1.240)$$

First we need to integrate:

$$\int dA^\mu(x)\exp\left[i\int d^4x\int d^4y\frac{e^2}{8}[A_\mu(x)A^\mu(y)+\frac{2}{e^2}\partial(y)\partial(x)]^2\delta(x-y)\right]. \tag{1.241}$$

For that we make the change of variable:

$$A_\mu(x)\Rightarrow A_\mu(x)-\partial_\mu(x). \tag{1.242}$$

This eliminates the dependence on the operator ∂^2 of the integral in Eq. (1.241) and leads to:

$$\begin{aligned}&\int dA^\mu(x)\exp\left[i\int d^4x\int d^4y\frac{e^2}{8}[A_\mu(x)A^\mu(y)+\frac{2}{e^2}\partial(y)\partial(x)]^2\delta(x-y)\right]=\\&\int dA^\mu\exp\left[i\int d^4x\int d^4y\frac{e^2}{8}\delta(x-y)[A_\mu(x)A^\mu(y)]^2\right]=d\end{aligned} \tag{1.243}$$

From Eqs. (1.238) and (1.243) we determine the partition function as:

$$Z=b+d\exp\Big[-i\frac{1}{2e^2}\partial^2(x)\partial^2(y)\delta(x-y)\Big]. \tag{1.244}$$

The Lorentz invariant two point function in the coordinate space is just:

$$\langle A^\mu(x)A_\mu(z)\rangle=\text{const}\int d^4y\frac{1}{Z}\frac{\delta(y-z)\delta Z}{\delta K(x,y)}\approx -i\frac{d}{2e^2(c+d)}K(x,z)=\text{const}K(x,z) \tag{1.245}$$

Eq. (1.245) leads to the following gauge field propagator in the Fourier space:

$$\text{Propagator}\approx\frac{p^2}{M^4}, \tag{1.246}$$

where M is an arbitrary scale.

Let us write the kinetic term in the Lagrangian associated with the propagator in Eq. (1.246):

$$\begin{aligned}&\int\frac{d^4p}{(2\pi)^4}A^\mu(-p)\frac{M^4}{p^2}A_\mu(p)=\\&\int\frac{d^4p}{(2\pi)^4}d^4xd^4yA^\mu(x)\exp[-ipx]A_\mu(y)\exp[ipy]\frac{M^4}{p^2}=\\&\int\frac{d^4p}{(2\pi)^4}d^4xd^4zA^a_\mu(x)\exp[ipz]A_\mu(z+x)\frac{M^4}{p^2}.\end{aligned} \tag{1.247}$$

Here we made the change of variables $y\to y-x$. Assume we express the last line in Eq. (1.247) in spherical coordinates and then we scale the momenta $pr=p'$ where $r=|\vec{z}|$ and

$p = |\vec{p}|$. The full integrand will gain a factor of r^2 indicating that we are dealing with a confining type of bilinear interaction instead of a regular kinetic term. Thus we claim that the actual particle kinetic term is a first indicator of a confining type of behavior.

The basic idea of the present work is that any field in a QFT Lagrangian no matter the spin has an alternative description in terms of fields of different spin although sometimes this substitution may be counterintuitive and complicated. In the case of an abelian Higgs model however there is a natural change of variables of both the Higgs and the abelian gauge field in terms of constituent fermions. Since the Higgs has 2 degrees of freedom and the gauge field at most 4 it is clear that bilinear combinations of one single fermion field (with 8 off-shell degrees of freedom) can accommodate both. We thus use this dual description and Fierz transformation of quadrilinear fermion terms to reexpress the initial Lagrangian in terms of possible rearrangements that can be made in it. The resulting Lagrangian contains a set of terms which are characterized by probabilities between zero and one. It turns out that the value of these probabilities exactly indicate the phase in which the system is in. Analogies and similarities with the standard treatments of the abelian Higgs model phase transitions in the literature are straightforward. The advantage of our approach is that it does not require any additional constraints or limitation of the Lagrangian as it was customary. By tuning these probabilities and using standard path integral methods we were able to determine the gauge propagator and the subsequent gauge kinetic term and thus illustrate the behavior of the theory in specific phases.

The phase structure of the abelian Higgs model is mostly known on a lattice, for fixed radial component of the scalar field or through numerical simulations. In the present work we do not employ any of these artifices to study the behavior of the system. Instead by using the dual description of the scalar and gauge fields in terms of fermion variables we extend the initial Lagrangian to take into account the various terms that might appear through a Fierz rearrangement of the fields. The most general Lagrangian obtained in this way is no longer gauge invariant but neither are all the phases in which the model is in.

By tuning the contributions of the interaction terms that are in the Lagrangian the system passes through three main phases which coincide exactly to those described in the literature: Higgs, confinement and Coulomb. However each phase has its own approximate Lagrangian which differs from one phase to another allowing one to determine the order parameter $\langle A^\mu(x)A_\mu(y)\rangle$ in the standard functional approach. There is also a fourth region in the parameter space where both the Higgs and confinement phases coexist showing that there is no clear distinction between the two of them.

What is particular to our approach besides the overall treatment is that the λ term in the Higgs potential plays no role whatsoever. The initial Lagrangian does not display spontaneous symmetry breaking (has the wrong mass term) but its overall modified structure leads to an actual Higgs phase.

The method is easy applicable to the pure abelian gauge model, electrodynamics without matter and may be extended to non abelian gauge invariant Lagrangians.

G. Phase transitions for nonabelian gauge theories.

Phase diagram of a nonabelian gauge theory with almost massless fermions has been the subject of many intensive studies over the past decades. The phase structure of supersymmetric QCD in terms of the number of flavors and colors has been elucidated by Seiberg and his collaborators. Although it is known that in QCD phases like confinement and chiral symmetry breaking are present in the phase diagram, their exact occurrence is far from being settled. Quite a few questions regarding the phase diagram of QCD are actual, especially when one makes a comparison to the supersymmetric counterpart.

In this subsection we shall adopt a new point of view with regard to the phase structure of QCD with fermions in the fundamental representation based on the knowledge that the potential between two sources is one of the indicators of the phase in which the theory is in. In order to establish the respective behavior we rely on the study of the Callan Symanzik equation for the two point gluon Green function. In this context we will be able to shed a new light on the possible phases and especially on the passage to the confinement and chiral symmetry breaking ones.

First let us briefly review what it is the current view with regard to the QCD phase diagram. We start from an $SU(N)$ gauge theories with N_f fermions in the fundamental representation. Beta function for this theory is known up to the fourth order. In the ' t Hooft scheme beta function stops at the first two orders coefficients. This part of the beta function is also renormalization scheme independent so provides a useful framework to study the behavior of the coupling constant. In our previous work by studying the global properties of the partition function it was proved that in a sense the two loop beta function,

$$\beta(g^2) = \frac{dg^2}{d\ln\mu^2} = -b\frac{g^4}{16\pi^2} - c\frac{g^6}{256\pi^4} =$$
$$-[\frac{11}{3}N - \frac{2}{3}N_f]\frac{g^4}{16\pi^2} - [\frac{34}{3}N^2 - 2\frac{N^2-1}{2N}N_f - \frac{10}{3}NN_f]\frac{g^6}{256\pi^4}. \tag{1.248}$$

contains the main properties of the gauge coupling constant. This is due to the fact that for any function of g^2 that expanded in g^2 respects the first two order renormalization scheme independent coefficients there is one renormalization scheme for which that function corresponds to a beta function for the gauge coupling constant. Here one makes the underlying assumption that a series expansion in g^2 makes sense and thus g^2 is small. For the nonperturbative regime or the limit between the perturbative and nonperturbative regions there are techniques that one may address. In the present work we adopt an approach based on the extension of the validity of the Callan-Symanzik equations to the nonperturbative domain. However some important indicators about the QCD phase diagram come from the beta function at two orders, no matter which renormalization scheme is chosen.

We observe that for $b > 0$, $N_f < \frac{11}{2}N$ the theory is asymptotically free which means that the coupling constant goes to zero for very large momenta. For $b < 0$ the theory loses its asymptotic freedom and it is infrared free. Since for $b < 0$ also $c < 0$ this property is preserved at two loops.

Let us discuss in more detail the region where $b > 0$. As one decreases the number of flavors from $N_f = \frac{11}{2}N$ first the coefficient $c < 0$. This means that there is a solution to the equation $\beta(g^2) = 0$ so the beta function has a non trivial fixed point. This occurs for ($\alpha = \frac{g^2}{4\pi}$):

$$-b\alpha^{*2} - \frac{c}{4\pi}\alpha^* = 0$$
$$\alpha^* = -\frac{4\pi b}{c}. \tag{1.249}$$

It can be shown that for N_f close to $\frac{11}{2}N$ the coupling constant is small and one can solve from the beta function for it to obtain that for small momenta the coupling constant approaches the fixed point. The corresponding phase is called the conformal phase and has been described in the literature . As N_f decreases even more the coupling constant increases and at some value of $N_f < \frac{11}{2}N$ confinement and chiral symmetry breaking occur. Although it is known that these two phases coexist for lower values of N_f it is still debatable if the phase transitions for both of them occur at the same critical value of N_f. We know that for SUSY QCD there are regions where the theory is confining but still chirally symmetric. Chiral symmetry breaking happens according to several studied when,

$$\alpha = \alpha_c = \frac{2\pi N}{3(N^2-1)}. \tag{1.250}$$

or when the anomalous dimension of the fermion mass operator $\gamma_m = 1$ (see the next sections for a discussion of this point).

One can further use $\alpha_c = \alpha^*$ to determine the critical number of flavors where chiral symmetry breaking takes place:

$$N_f^c = N\frac{100N^2 - 66}{25N^2 - 15}. \tag{1.251}$$

Thus for $N_f^c < N_f < \frac{11}{2}N$ the theory is in the conformal phase; has a fixed point, is chirally symmetric and deconfined. For $N_f < N_f^c$ the theory is confined and with the chiral symmetry broken.

It was shown that in SUSY QCD there is a window $N_c+1 < N_f < 3N_c$ where the theory is asymptotically free but not confining. Also for $N_f = N_c+1$ the theory displays confinement but not chiral symmetry breaking. In the following we will claim that as opposed to SUSY QCD in QCD chiral symmetry breaking takes place before confinement.

We shall start from the Callan Symanzik equation for the two point gluon Green function:

$$[p\frac{\partial}{\partial p} + 2 - \beta(g)\frac{\partial}{\partial g} - 2\gamma_3 - \gamma_m m\frac{\partial}{\partial m}]G^2(p,m,g) = 0. \tag{1.252}$$

Here p is the momentum, $\beta(g)$ is the beta function, γ_3 is the anomalous dimension of the gluon wave function and γ_m is the anomalous dimension of the fermion mass operator. We need to clarify what are the exact definitions of the beta function and anomalous dimension that should be introduced in Eq. (1.252). We will work in the background gauge field method where Z_3 is the renormalization constant for the gluon wave function. Then:

$$\beta(g) = \frac{1}{2}g\frac{\partial Z_3}{\partial \ln\mu}$$
$$\gamma_3 = -\frac{1}{2}\frac{\partial Z_3}{\partial \ln\mu} = -\frac{\beta(g)}{g}. \tag{1.253}$$

For the correct definiton of the other anomalous dimensions that enter the Callan Symanzik equation one takes into account the positive shift of the corresponding quantity as opposed to the standard dimensional regularization definition where one takes into account the negative one. Thus the correct expression for γ_m in Eq. (1.252) is,

$$\gamma_m = -\frac{1}{m}\frac{\partial m}{\partial \ln \mu}. \tag{1.254}$$

We are interested in the behavior of the two point Green function at low energy. Without any loss of generality we can take $p = km$ where p is the momentum, m is the fermion mass (we consider a generic mass for all fermions) and k is an adimensional scaling constant. Then the Callan Symanzik equation (1.252) becomes:

$$[p\frac{\partial}{\partial p}(1-\gamma_m) + 2 - \beta(g)\frac{\partial}{\partial g} - 2\gamma_3]G^2(p,g,m) = 0. \tag{1.255}$$

In the background gauge field method (see Eq. (1.295)) this can further be simplified to:

$$[p\frac{\partial}{\partial p}(1-\gamma_m) + 2 - \beta(g)(\frac{\partial}{\partial g} - \frac{2}{g})]G^2(p,g,m) = 0. \tag{1.256}$$

We thus have a differential equation dependent on two variables p and g and we expect critical behavior when the different coefficients of the differential operators change the sign. First we need a solution of the two point function of the Eq. (1.256) at the infrared fixed point. We thus ask $\beta(g) = 0$ (for which $\frac{g^{*2}}{4\pi} = -\frac{4\pi b}{c}$) which leads to:

$$[(1-\gamma_m)p\frac{\partial}{\partial p} + 2]G^2(p,g,m) = 0. \tag{1.257}$$

This equation has a simple solution:

$$G^2(p,g,m) \approx \frac{1}{p^{\frac{2}{1-\gamma_m}}}. \tag{1.258}$$

We require $\gamma_m = 2$, where at one loop:

$$\gamma_m = 3\frac{N^2-1}{N}\frac{g^2}{16\pi^2}, \tag{1.259}$$

and solve for the corresponding value g_c:

$$\frac{g_c^2}{4\pi} = \frac{8\pi N}{3(N^2-1)}. \tag{1.260}$$

Around this particular value for γ_m the Green function $G^2(p,g,m)$ will be close to:

$$G^2(p,g,m) \approx p^2, \tag{1.261}$$

which for low momenta and in the coordinate space will lead to a confining potential of the type $V(r) \approx r$. We claim that this is a clear indicator that the transition to the confinement phase takes place around this point. Since we are already are in the critical regime where $g = g^*$ we further require $g_c = g^*$ to obtain the corresponding number of flavors:

$$N_f^c = N\left(\frac{101N^2 - 33}{32N^2 - 12}\right). \tag{1.262}$$

Note that this is in slight contradiction with previous studies (see section I) which claim that the transition to confinement take place at the value in Eq. (1.251).

Early studies suggest that the behavior of the mass around the chiral symmetry breaking point is:

$$m(\mu) \approx \frac{1}{\mu} \tag{1.263}$$

which in our notation means $\gamma_m = -\frac{d\ln m}{d\ln\mu} = 1$. For γ_m approaching 1 from above we observe from Eq. (1.257) that the Green function becomes zero for low momenta and infinite for high momenta indicating again a critical behavior. We claim in agreement with early studies that this behavior of the Green function indicates the transition to the chiral symmetry breaking phase. The condition $\gamma_m = 1$ leads to the critical coupling constant α_s (g_s):

$$\alpha_s = \frac{g_s^2}{4\pi} = \frac{4\pi N}{3(N^2 - 1)}. \tag{1.264}$$

Furthermore the constraint $g_s = g^*$ yields the critical number of flavors for the transition to the chiral symmetry breaking phase:

$$N_f^s = N\left(\frac{67N^2 - 33}{19N^2 - 9}\right). \tag{1.265}$$

Note that this value is still in the region where $b > 0$ and $c < 0$ where the non trivial fixed point exists.

For N large the transition to the chiral symmetry breaking phase happens at $N_f^s \approx 3.5N$ whereas the confinement transition occurs at $N_f^c \approx 3.1N$ showing that for certain values of N there are values of N_f for which, as opposed to the supersymmetric case, there is chiral symmetry breaking but no confinement. For example for $N = 3$, $N_f^c = 9.5$ and $N_f^s = 10.5$ and for $N = 4$, $N_f^c = 12.7$ and $N_f^s = 14.08$ thus proving our point. More specific for $N = 3$ and $N_f = 10$ there is chiral symmetry breaking and no confinement as confinement sets in at $N_f = 9$. Also for $N = 4$, $N_f = 14$ or $N_f = 13$ there is chiral symmetry breaking and no confinement as the latter commences at $N_f = 12$.

To resume we showed that for an $SU(N)$ abelian gauge theory with fermions in the fundamental representation confinement sets in for slightly lower values of N_f than chiral symmetry breaking and thus there are region in the phase diagram in terms of N and N_f where there is chiral symmetry breaking but no confinement. This result can be related to opposite findings in supersymmetric QCD.

According to our analysis the two point Green function of the gluon suggest that in the region $b > 0$ and $c < 0$ there are only two main phase transitions associated to confinement and chiral symmetry breaking. For $b < 0$ the system is in the free infrared phase. In this section we will try to find if there are signs of critical behavior in the region $b > 0$, $c \geq 0$. For that we rewrite the Callan Symanzik equation in the background gauge field method:

$$[p\frac{\partial}{\partial p}(1-\gamma_m) + 2 - \beta(g)(\frac{\partial}{\partial g} - \frac{2}{g})]G^2(p,g,m) = 0, \tag{1.266}$$

and we look for possible changes of sign in the differential operators in Eq. (1.266) for $\beta(g) \neq 0$. We have two possibilities:

$$\begin{aligned}
&(\frac{\partial}{\partial g} - \frac{2}{g})G^2(p,g,m) = 0 \\
&(2 + \frac{2}{g}\beta(g))G^2(p,g,m) = 0.
\end{aligned} \tag{1.267}$$

The constraint in the first line of Eq. (1.267) is in the perturbative regime and does not bring anything new. The condition in the second line leads to $\beta(g) = -g$ and to the simplified equation:

$$[(1-\gamma_m)\frac{\partial}{\partial p} - \beta(g)\frac{\partial}{\partial g}]G^2(p,g,m) = 0. \tag{1.268}$$

This equation can be reduced to:

$$p\frac{\partial G^2(p,g,m)}{\partial p} = \frac{g}{\gamma_m - 1}\frac{\partial G^2(p,g,m)}{\partial g}. \tag{1.269}$$

There is no much we can learn from the behavior of the Green function from the above equation (as it is dependent on the constants of integration) but we can spot a singularity for $\gamma_m = 1$ which might indicate critical behavior. Since we were already in the regime with confinement and chiral symmetry breaking this possible critical behavior might indicate a transition to a phase or region where these two become strong. The relations:

$$\begin{aligned}
&\beta(g) = -g \\
&\gamma_m = 1,
\end{aligned} \tag{1.270}$$

lead to the corresponding critical number of flavors:

$$N_f^{cs} = \frac{40N^4 + 21N^2 - 27}{19N^3 - 9N}. \tag{1.271}$$

However this tentative phase transition should be confirmed by alternative studies.

Phase diagram at zero temperature of a $SU(N)$ gauge theory with fermions in different representations has been discussed extensively in the literature. In particular the critical number of flavors at which chiral symmetry breaking happens has been obtained using many approaches like the gap equation, nonperturbative effective potential , nonperturbative dynamics of the gauge fields , lattice calculations , Nambu Jona Lasinio model, Bethe Salpeter equation. In most of these cases the critical number of flavors obeys $N_f^c < 4N$ thus suggesting that many of the methods converge to a similar result.

If the regions of confinement and chiral symmetry breaking are confirmed than the rest of the phases that exist for $N_f > N_f^c$ are already settled and are determined entirely by the behavior of the beta function. Thus the infrared free phase corresponds to $N_f > \frac{11}{2}N$ where the theory loses its asymptotic freedom and the conformal window belongs to the region $N_f^c < N_f < \frac{11}{2}N$.

In this work we introduced a new way to study some of the features of the phase diagram in the (N, N_f) plane that relies only partially on the beta function. We start from what one can learn from the beta function and anomalous dimensions and then we use this knowledge to analyze the two point gluon Green function from the corresponding Callan Symanzik equation. We argue that these equations describe accurately the all order Green functions and thus contain hints about their nonperturbative behavior. Our approach is particularly useful to describe to transitions to the confinement and chiral symmetry breaking phase because we have some information about how the potential between two sources should behave for these cases. One of our main results is that that the transitions to chiral symmetry breaking should occur for,

$$N_f^s = N\left(\frac{67N^2 - 33}{19N^2 - 9}\right), \tag{1.272}$$

whereas transition to the confinement phase happens for a slightly lower value for N_f:

$$N_f^c = N\left(\frac{101N^2 - 33}{32N^2 - 12}\right). \tag{1.273}$$

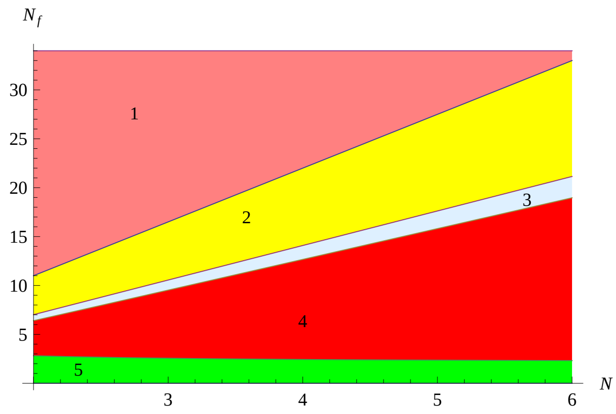

FIG. 1: Phase diagram of an $SU(N)$ gauge theory in terms of the number of colors N and flavors N_f. The numbers from 1-5 represent the distinct phases:1-infrared free; 2-conformal; 3-weak chiral symmetry breaking; 4-confinement and chiral symmetry breaking; 5-strong confinement and chiral symmetry breaking. Note that chiral symmetry breaking phase corresponds to region 3 but all subsequent confinement phases are also characterized by chiral symmetry breaking.

Furthermore the study of the Callan Symanzik equation suggest that for values of N_f even lower there is a possibility that a new phase transition occurs:

$$N_f^{cs} = \frac{40N^4 + 21N^2 - 27}{19N^3 - 9N}. \tag{1.274}$$

We suggest that this new phase transition corresponds to a region where both confinement and symmetry breaking become strong as the two loop beta function $\frac{\beta(g)}{g} \leq -1$. However we could not extract more information with regard to the behavior of the system in this phase from the analysis we made. A complete confirmation and description of this phase would require alternative nonperturbative techniques.

In Fig 1 we compile all the information in the present work and depict the full zero temperature phase diagram of a $SU(N)$ gauge theory with fermions in the fundamental representation.

H. Partition function for a scalar theory with spontaneous symmetry breaking

In the context in which the Higgs boson of the standard model has been at the forefront of both theoretical and experimental search for decades and the corrections to the Higgs boson mass or more exactly the issues associated to them have triggered a flurry of theoretical papers it is important to study the problems associated with scalars and their masses in smaller set-ups or more amenable models.

In this subsection we extend the approach employed in the previous subsections to study the two point correlator and the corrections to the Higgs mass in an abelian Higgs model with spontaneous symmetry breaking.

We consider the abelian Higgs model with the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^{\mu}\Phi)^{\dagger}(D_{\mu}\Phi) - V(\Phi), \tag{1.275}$$

where:

$$\begin{aligned} &D_{\mu} = \partial_{\mu} + ieA_{\mu} \\ &V(\Phi) = -\mu^2\Phi^*\Phi + \frac{\lambda}{2}(\Phi^*\Phi)^2. \end{aligned} \tag{1.276}$$

This model displays spontaneous symmetry breaking at the minimum of the potential:

$$\Phi_0^2 = v^2 = \left(\frac{\mu^2}{\lambda}\right). \tag{1.277}$$

We expand the field around its vev to get:

$$\Phi(x) = v + \frac{1}{\sqrt{2}}(h(x) + i\phi(x)), \tag{1.278}$$

where $\phi(x)$ is the Goldstone boson. We shall work in the R_{ξ} gauges with $\xi = 1$ where the gauge fixed Lagrangian has the form:

$$\begin{aligned} &\mathcal{L} = -\frac{1}{2}A_{\mu}(-g^{\mu\nu}\partial^2 + e^2v^2g^{\mu\nu})A_{\nu} + \\ &\frac{1}{2}(\partial_{\mu}h)^2 - \frac{1}{2}(2\lambda v^2)h^2 + \frac{1}{2}(\partial_{\mu}\phi)^2 - (ev)^2\phi^2 + \\ &e^2v^2A_{\mu}A_{\nu}g^{\mu\nu} + \frac{2}{\sqrt{2}}e^2vhA_{\mu}A_{\nu}g^{\mu\nu} + \frac{e^2}{2}h^2A_{\mu}A_{\nu}g^{\mu\nu} - \\ &(-\frac{\mu^4}{2\lambda} + \frac{1}{8}\lambda h^4 + \frac{1}{8}\lambda\phi^4 + \frac{\lambda}{\sqrt{2}}vh^3 + \frac{\lambda}{4}h^2\phi^2 + \frac{\lambda}{\sqrt{2}}hv\phi^2) + \\ &\bar{c}(-\partial^2 + 2(ev)^2(1 + \frac{h}{v}))c. \end{aligned} \tag{1.279}$$

The last term in Eq. (1.279) corresponds to the ghost term of the gauge fixed Lagrangian.

We shall write only the quadratic term of the space time integral of the Lagrangian $\mathcal{L}$ on lattice with volume V in the Fourier space:

$$\begin{aligned}\int d^4x\mathcal{L}_2 = \quad & -\frac{1}{2}\frac{1}{V}\sum_n A_\mu(p_n)g^{\mu\nu}(p_n^2-m_A^2)A_\nu(-p_n)+\\ & \frac{1}{2}\frac{1}{V}\sum_n h(p_n)(p_n^2-m_h^2)H(-p_n)+\\ & \frac{1}{2}\frac{1}{V}\sum_n \phi(p_N)(p_n^2-m_A^2)\phi(-p_n)+\\ & \frac{1}{V}\sum_n \bar{c}(p_n)(p_n^2-m_A^2)c(-p_n).\end{aligned} \tag{1.280}$$

We start with the expression for the two point Higgs scalar function in the path integral formalism:

$$\langle\Omega|Th(x_1)h(x_2)|\Omega\rangle = \lim_{T\to\infty(1+i\varepsilon)}\frac{\int dA_\mu(x)dh(x)d\phi(x)d\bar{c}(x)dc(x)h(x_1)h(x_2)\exp[i\int d^4x\mathcal{L}]}{\int dA_\mu(x)dh(x)d\phi(x)d\bar{c}(x)dc(x)\exp[i\int d^4x\mathcal{L}]}. \tag{1.281}$$

We first write in the Fourier space:

$$h(x_1)h(x_2) = \frac{1}{V^2}\sum_m \exp[-ik_mx_1]h(k_m)\sum_l \exp[-ik_lx_2]h(k_l), \tag{1.282}$$

and furthermore:

$$\begin{aligned}&\langle\Omega|Th(x_1)h(x_2)|\Omega\rangle = \frac{1}{V^2}\sum_{m,l}\exp[-ik_mx_1-ik_lx_2]\times\\ &\lim_{T\to\infty(1+i\varepsilon)}\frac{\prod_{n,p,r,s,t}\int dA_\mu(k_n)dh(k_p)d\phi(k_r)d\bar{c}(k_s)dc(k_t)h(k_m)h(k_l)\exp[i\int d^4x\mathcal{L}]}{\prod_{n,p,r,s,t}\int dA_\mu(k_n)dh(k_p)d\phi(k_r)d\bar{c}(k_s)dc(k_t)\exp[i\int d^4x\mathcal{L}]},\end{aligned} \tag{1.283}$$

where the exponent should be expressed also in the Fourier space.

Next we consider the function:

$$\begin{aligned}&I_{x-y} = \frac{1}{V}\sum_m \exp[-ik_m(x_1-x_2)]\langle h(k_m)h(-k_m)\rangle =\\ &\frac{1}{V}\sum_m \exp[-ik_m(x_1-x_2)]\times\\ &\lim_{T\to\infty(1+i\varepsilon)}\frac{\prod_{n,p,r,s,t}\int dA_\mu(k_n)dh(k_p)d\phi(k_r)d\bar{c}(k_s)dc(k_t)h(k_m)h(-k_m)\exp[id^4x\mathcal{L}]}{\prod_{n,p,r,s,t}\int dA_\mu(k_n)dh(k_p)d\phi(k_r)d\bar{c}(k_s)dc(k_t)\exp[id^4x\mathcal{L}]}\end{aligned} \tag{1.284}$$

The relation between Eq. (1.283) and Eq. (1.284) is given by:

$$
\begin{aligned}
&I_{x-y} = \frac{1}{V}\sum_m \exp[-ik_m(x_1-x_2)]\langle h(k_m)h(-k_m)\rangle = \\
&\frac{1}{V}\sum_m \exp[-ik_m(x_1-x_2)]\int d^4z_1 \exp[ik_m z_1]\int d^4z_2 \exp[-ik_m z_2]\langle h(z_1)h(z_2)\rangle = \\
&\frac{1}{V}\sum_m \int d^4z_1 d^4z_2 \exp[-ik_m(x_1-x_2-z_1+z_2)]\langle h(z_1)h(z_2)\rangle = \\
&\int d^4z_1 d^4z_2 \delta(x_1-x_2-z_1+z_2)\langle h(z_1)h(z_2)\rangle = \\
&\int d^4z_2\langle h(z_2+x_1-x_2)h(z_2)\rangle.
\end{aligned}
\tag{1.285}
$$

But according to the definition of the two point function the following relation holds:

$$
\begin{aligned}
&\langle h(z_2+x_1-x_2)h(z_2)\rangle = \int \frac{d^4p}{(2\pi)^4} exp[-ip(x_1-x_2)]\frac{1}{p^2-m^2-M(p^2)} = \\
&\langle\Omega|Th(x_1)h(x_2)|\Omega\rangle,
\end{aligned}
\tag{1.286}
$$

where $M(p^2)$ is the all order correction to the two point function in the Fourier space of the scalar h. From Eqs. (1.285) and (1.286) we then infer on the lattice:

$$
\begin{aligned}
&I_{x_1-x_2} = \int d^4z_2\langle\Omega|Th(x_1)h(x_2)|\Omega\rangle = \\
&V\int \frac{d^4p}{(2\pi)^4} exp[-ip(x_1-x_2)]\frac{1}{p^2-m^2-M(p^2)},
\end{aligned}
\tag{1.287}
$$

where V is the volume of space time. Here it is understood that:

$$
\frac{1}{V}\sum_m \to \int \frac{d^4p}{(2\pi)^4}
\tag{1.288}
$$

in the continuum limit.

This shows that in order to determine the two point function for the Higgs boson it is sufficient to evaluate $I_{x_1-x_2}$. In order to do that we first separate from the Lagrangian the quadratic part as in Eq.(1.280). We then write:

$$
\begin{aligned}
&\langle h(k_m)h(-k_m)\rangle = \\
&\frac{-2iV\frac{\delta}{\delta k_m^2}\prod_{n,p,r,s,t}\int dA_\mu(k_n)dh(k_p)d\phi(k_r)d\bar{c}(k_s)dc(k_t)\exp[i\int d^4x\mathcal{L}]}{\prod_{n,p,r,s,t}\int dA_\mu(k_n)dh(k_p)d\phi(k_r)d\bar{c}(k_s)dc(k_t)\exp[i\int d^4x\mathcal{L}]} - \\
&-\langle\phi(k_m)\phi(-k_m)\rangle + 2\langle c(k_m)\bar{c}(-k_m)\rangle + \langle A_\mu(k_m)A^\mu(-k_m)\rangle,
\end{aligned}
\tag{1.289}
$$

where the quantities in the brackets are defined similarly with the definition for the Higgs boson.

One can compute the full partition function (in which no additional measure of integration is introduced) to extract the dependence on the momenta:

$$Z = \text{const} \prod_n (\frac{iV}{p_n^2 - m_h^2})^{1/2} (\frac{-iV}{p_n^2 - m_A^2})^2 (\frac{iV}{p_n^2 - m_A^2})^{1/2} iV(p_n^2 - m_A^2), \tag{1.290}$$

where the first factor corresponds to the Higgs boson, the second to the gauge boson, the third to the Goldstone boson and the last one to the ghosts. The Eqs. (1.289) and (1.290) lead to:

$$\begin{aligned} \frac{Vi}{p^2 - m_h^2 - M(p^2)} &= \frac{Vi}{p^2 - m_h^2} + \frac{3Vi}{p^2 - m_A^2} - \\ \frac{iV}{p^2 - m_1^2 - \Sigma_1(p^2)} &+ \frac{2iV}{p^2 - m_2^2 - \Sigma_2(p^2)} - \frac{4iV}{p^2 - m_A^2 - \Sigma_A(p^2)}. \end{aligned} \tag{1.291}$$

Here all the masses are bare masses and the quantities $M(p^2)$, $\Sigma_A(p^2)$, $\Sigma_1(p^2)$ and $\Sigma_2(p^2)$ are the corrections to the two point functions for the Higgs boson, gauge boson, Goldstone boson and ghost respectively. We can further write Eq. (1.291) as:

$$\begin{aligned} \frac{1}{p^2 - m_h^2} + \frac{3}{p^2 - m_A^2} &= \frac{1}{p^2 - m_h^2 - M(p^2)} + \frac{1}{p^2 - m_1^2 - \Sigma_1(p^2)} - \\ \frac{2}{p^2 - m_2^2 - \Sigma_2(p^2)} &+ \frac{3}{p^2 - m_A^2 - \Sigma_A(p^2)}. \end{aligned} \tag{1.292}$$

Here in writing the right hand side of Eq. (1.292) we took into account two important facts:

1) The actual bare masses ξm_A^2 for the Goldstone boson and the ghost is gauge dependent and both the gauge parameter and the mass can receive quantum correction.

2) The full propagator in the Feynman gauge has apart from the correction to the mass the same expression as the bare one since in the model there are no derivative interactions pertaining to the gauge boson.

The full propagator for a boson particle X has a pole of the form:

$$\text{Propagator} \approx \frac{1}{p^2 - m_X^2 - \Sigma_X(p^2)}, \tag{1.293}$$

where m_X is the bare mass and Σ_X is the all order correction to the two point function. It is known that this propagator has a pole at the physical mass of the particle and also weaker singularities in the form of branch cuts. We expand the denominator in the right hand side of Eq. (1.293) to obtain:

$$p^2 - m_X^2 - \Sigma_X(p^2) =$$
$$p^2 - m_X^2 - \Sigma_X(p^2 = m_{Xphys}^2) - \Sigma'(p^2)|_{p^2=m_{Xphys}^2}(p^2 - m_{Xphys}^2) - \Sigma''(p^2)|_{p^2=m_{Xphys}^2}(p^2 - m_{Xphys}^2)^2 ... =$$
$$(p^2 - m_{Xphys}^2)(1 - \Sigma'(p^2)|_{p^2=m_{Xphys}^2} - \Sigma''(p^2)|_{p^2=m_{Xphys}^2}(p^2 - m_{Xphys}^2)...) =$$
$$(p^2 - m_{Xphys}^2)S_X(p^2) - \Sigma''(p^2)|_{p^2=m_{Xphys}^2}(p^2 - m_{Xphys}^2) - ..., \tag{1.294}$$

where we define:

$$m_X^2 + \Sigma_X(p^2 = m_{Xphys}^2) = m_{Xphys}^2$$
$$(1 - \Sigma'(p^2)|_{p^2=m_{Xphys}^2}) = S_X = \frac{1}{Z}, \tag{1.295}$$

and $S_X(m_{Xphy}^2) = \frac{1}{Z} = 1$ (the residue of the propagator at the pole is equal to 1) by the renormalization condition. In consequence:

$$\frac{1}{p^2 - m_X^2 - \Sigma_X(p^2)} = \frac{Z}{p^2 - m_{Xphys}^2} + \text{regular terms}, \tag{1.296}$$

and there are no other poles on the right hand side of Eq. (1.296) since contributions from two or more particles intermediate states do not produce other poles.

The same expansion can apply to all the quantities on the right hand side of Eq. (1.292) and leads to:

$$\frac{1}{p^2 - m_h^2} + \frac{3}{p^2 - m_A^2} = \frac{1}{(p^2 - m_{hphys}^2)} + \frac{1}{(p^2 - m_{1phys}^2)} -$$
$$\frac{2}{(p^2 - m_{2phys}^2)} + \frac{4}{(p^2 - m_{Aphys}^2)} + \text{terms regular.} \tag{1.297}$$

Then by comparing the pole structure on the right hand side and left hand side of the Eq. (1.297) and considering the fact that the quantities S_A, S_1, S_2 have no zeroes we conclude that:

$$m_h^2 = m_{hphys}^2$$
$$m_A^2 = m_{Aphys}^2$$
$$m_{1phys}^2 = m_{Aphys}^2 = m_A^2$$
$$m_{2phys}^2 = m_{Aphys}^2 = m_A^2 \tag{1.298}$$

Note that at first glance there are other possible combinations of results but by considering the possible gauge dependence and the fact that the results should be correct no matter the gauge chosen and also the residue structure for this case it is clear that the possibility outlined in Eq. (1.298) is the only choice. A final remark is in order: these results cannot be generalized easily and may lead to different result for a nonabelian gauge theory due to the presence of the derivative terms in the Lagrangian.

I. Applications of the new method to the standard model of elementary particles

The standard model of elementary particles has received its most important experimental confirmation with the discovery of the Higgs boson by the Atlas and CMS experiments. One of the most important theoretical issue associated with the standard model Higgs boson is that of naturalness which stems from the existence of significant quadratic corrections to the Higgs boson mass. In order to predict the mass of the Higgs boson one needs additional assumptions or hypothesis as the mass of the Higgs boson cannot be extracted only from the standard model parameters. In this work we rely on the properties of the standard model Lagrangian and partition function before and after spontaneous symmetry breaking to make the assumption that the gauge invariant operators are conserved at the transition between the symmetric and spontaneously broken phases. Based on this we calculate at one loop quantum correlators to which we add the most important two loop contributions to determine a mass of the Higgs boson very close to the central experimental value. This is done by relating the quadratic corrections to the gauge invariant kinetic term for the Higgs doublet for the symmetric and spontaneously broken Lagrangians.

We consider the electroweak sector of the standard model given by the Lagrangian:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4, \tag{1.299}$$

where,

$$\mathcal{L}_1 = -\frac{1}{4}F^i_{\mu\nu}F^{\mu\nu i} - \frac{1}{4}G^{\mu\nu}G_{\mu\nu}, \tag{1.300}$$

with,

$$\begin{aligned}
F^i_{\mu\nu} &= \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk}A^j_\mu A^k_\nu \\
G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu.
\end{aligned} \tag{1.301}$$

Furthermore,

$$\mathcal{L}_2 = \bar{\Psi} i \gamma^\mu D_\mu \Psi. \tag{1.302}$$

Here we used a generic notations for all fermion terms and for example:

$$D_\mu \Psi = (\partial_\mu - ig\vec{T}A_\mu - ig'\frac{Y}{2}B_\mu)\Psi. \tag{1.303}$$

Next,

$$\begin{aligned}
\mathcal{L}_3 &= (D^\mu \Phi)^\dagger (D_\mu \Phi) - V(\Phi) \\
\mathcal{L}_4 &= f_e \bar{l}_L \Phi e_R + f_u \bar{q}_L \tilde{\Phi} u_R + f_d \bar{q}_L \Phi d_R + h.c.
\end{aligned} \tag{1.304}$$

Here Φ is the standard model Higgs doublet,

$$\Phi = \begin{pmatrix} \Phi^+ \\ \Phi_0 \end{pmatrix}, \tag{1.305}$$

with,

$$\begin{aligned}
D_\mu \Phi &= (\partial_\mu - \frac{i}{2} g \vec{\tau} \vec{A}_\mu - \frac{i}{2} g' B_\mu)\Phi \\
V(\Phi) &= -m_0^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 \\
\tilde{\Phi} &= i\tau_2 \Phi^*.
\end{aligned} \tag{1.306}$$

The potential in Eq. (1.306) displays spontaneous symmetry breaking according to the structure:

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi'^+ \\ v + h + i\Phi_3, \end{pmatrix} \tag{1.307}$$

where $v = \frac{m_0^2}{\lambda}$ is the vacuum expectation value, h is the Higgs and Φ_3, $\Phi'^\pm$ are the Goldstone bosons. As a result of spontaneous symmetry breaking the mass of the Higgs boson will become $2m_0^2$ whereas the Goldstone bosons will remain massless.

We are mainly interested in the gauge kinetic term for the Higgs boson. For future use we give below the detailed expression for this term as a function of the mass eigenstates of the gauge fields after spontaneous symmetry breaking:

$$\begin{aligned}
&(D^\mu\Phi)^\dagger(D_\mu\Phi) = \partial^\mu\Phi^-\partial_\mu\Phi^+ + \partial^\mu\Phi_0^*\partial_\mu\Phi_0 + \\
&\partial^\mu\Phi^-[-\frac{i}{2}(bZ_\mu + cA_\mu)\Phi^+ - \frac{i}{\sqrt{2}}gW_\mu^+\Phi_0 g] + \\
&\partial_\mu\Phi^+[\frac{i}{2}(bZ^\mu + cA^\mu)\Phi^- + \frac{i}{\sqrt{2}}gW^{\mu-}\Phi_0 g] + \\
&\partial^\mu\Phi_0^*[-\frac{i}{\sqrt{2}}gW_\mu^-\Phi^+ - \frac{i}{2}aZ_\mu\Phi_0] + \\
&\partial_\mu\Phi_0[\frac{i}{\sqrt{2}}gW^{\mu+}\Phi^- + \frac{i}{2}aZ^\mu\Phi_0^*] + \\
&[\frac{1}{2}(bZ^\mu + cA^\mu)\Phi^- + \frac{1}{\sqrt{2}}gW^{\mu-}\Phi_0][\frac{1}{2}(bZ_\mu + cA_\mu)\Phi^+ + \frac{1}{\sqrt{2}}gW_\mu^+\Phi_0] + \\
&[\frac{1}{\sqrt{2}}W^{\mu-}\Phi^+ + \frac{a}{2}Z^\mu\Phi_0][\frac{1}{\sqrt{2}}W_\mu^+\Phi^- + \frac{a}{2}Z_\mu\Phi_0].
\end{aligned} \tag{1.308}$$

Here we made the notations:

$$\begin{aligned}
a &= \sqrt{g^2 + g'^2} \\
b &= \frac{g^2 - g'^2}{\sqrt{g^2 + g'^2}} \\
c &= \frac{2gg'}{\sqrt{g^2 + g'^2}} = 2e.
\end{aligned} \tag{1.309}$$

The gauge kinetic term for the Higgs boson after spontaneous symmetry breaking can be read off very easily by applying Eq. (1.307) to Eq. (1.308).

A gauge invariant Lagrangian needs fixing and one can show that the corresponding partition function remains unchanged through this procedure up to an irrelevant proportionality factor. In this work we shall make the assumption that the vacuum correlators of the gauge invariant operators are exactly the same before and after spontaneous symmetry breaking. In other words the gauge invariant quantum correlator calculated for the symmetric partition function that contains tachyonic states is equal to the corresponding quantity calculated for the partition function with spontaneously broken symmetry. This assumption is justified by the fact that in the actual partition function the transition from one state to another can be simply expressed in a change of variable in the Higgs sector $\Phi_0 = \frac{1}{\sqrt{2}}(v + h + i\Phi_3)$. In order for this equivalence of the quantum correlators to be valid it is necessary to keep, besides this change of variable, the partition function unchanged before and after spontaneous symmetry breaking. This implies that the gauge fixing of the Lagrangian must be done similarly and consistent in both cases without the introduction of additional terms in the Lagrangian. The most amenable gauge is the Landau gauge $\xi = 0$ which will be used before and after spontaneous symmetry breaking with the amend that the gauge fixing function must be the same in both situations. This a slight depart from the R_ξ gauges for theories with spontaneous symmetry breaking where for example the gauge function for a generic case is if the type $G = \frac{1}{\sqrt{\xi}}(\partial^\mu A_\mu - \xi v \phi)$ where ϕ is the Goldstone boson associated to the gauge field A_μ. Here we shall take simply $G = \frac{1}{\sqrt{\xi}}(\partial^\mu A_\mu)$ for both cases.

In the standard model Lagrangian there are quite a few of gauge invariant operators but we are interested in those that contain the Higgs doublet and no constant terms. The most relevant among these is the gauge kinetic term for the Higgs doublet. We shall equate the quadratic corrections at one loop before and after spontaneous symmetry breaking. Before spontaneous symmetry breaking this term reads:

$$(D^\mu \Phi^\dagger)(D_\mu \Phi) = \partial^\mu \Phi^- \partial_\mu \Phi^+ + \partial^\mu \Phi_0^* \partial_\mu \Phi_0 + \text{trilinear and quadrilinear terms} \quad (1.310)$$

whereas after spontaneous symmetry breaking:

$$\begin{aligned}
&(D^\mu \Phi^\dagger)(D_\mu \Phi) = \\
&\frac{1}{2}\partial^\mu h \partial_\mu h + \frac{1}{2}\partial^\mu \Phi'^- \partial_\mu \Phi'^+ + \frac{1}{2}\partial^\mu \Phi_3 \partial_\mu \Phi_3 + \\
&-m_Z^2 Z^\mu \partial_\mu \Phi_3 - i\frac{1}{\sqrt{2}} W^{\mu +}\partial_\mu \Phi'^- + i\frac{1}{\sqrt{2}} W^-_\mu \partial^\mu \Phi'^+ + \\
&\text{trilinear and quadrilinear terms.}
\end{aligned} \quad (1.311)$$

Next we compute the quadratic corrections corresponding to Eq. (1.310):

$$A = \langle \int d^4x (D^\mu \Phi^\dagger)(D_\mu \Phi)\rangle = \delta(0) \int_{|\vec{p}| \le m_0} \frac{d^4p}{(2\pi)^4} i \frac{p^2}{p^2 + m_0^2 + i\epsilon} \frac{1}{2} \times 4, \quad (1.312)$$

where the factor 4 comes from the four scalars with the same mass in the Higgs doublet (see Appendix A for the choice of the tachyon propagator and for how the tachyon contribution is treated). Similarly after spontaneous symmetry breaking the same quantum correlator yields:

$$\begin{aligned}
&B = \langle (D^\mu \Phi^\dagger) D_\mu \Phi)\rangle = \\
&\delta(0) \int \frac{d^4p}{(2\pi)^4} [\frac{i}{2}\frac{p^2}{p^2 - m_h^2} + \frac{3i}{2}\frac{p^2}{p^2} - i\frac{3}{2} m_Z^2 \frac{1}{p^2 - m_Z^2} - 3i \frac{1}{p^2 - m_W^2}].
\end{aligned} \quad (1.313)$$

The calculations are done as usual by rotating to the euclidean space. We use,

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m_X^2} = -\frac{i}{16\pi^2}[\Lambda^2 - m_X^2 \ln[\frac{\Lambda^2}{m_X^2}]], \quad (1.314)$$

to find by equating the quadratic divergences before and after spontaneous symmetry breaking (see also Appendix A):

$$-2m_0^2 = -\frac{3}{2}m_Z^2 - 3m_W^2 + \frac{m_h^2}{2} \tag{1.315}$$

Then by assuming the bare masses $m_h^2 = 2m_0^2$ we determine:

$$m_h^2 = m_Z^2 + 2m_W^2. \tag{1.316}$$

Eq. (1.316) leads, by introducing the physical masses for the gauge boson to an estimate $m_h \approx 145.7$ GeV which is very large compared to the known physical value of the Higgs boson.

However this result is a gross approximation for several reasons. First the quadratic term that relates the Goldstone boson with the gauge boson contributes significantly at one loop. Second the two loop contributions especially those that involve the top quark and the tadpole diagrams give important quadratic corrections at two loop which alter substantially the one loop result. Next one should consider also the one loop corrections to the bare mass in order to extract a suitable mass for the Higgs boson since these corrections contains quadratic divergences. With one amend that we will state later we shall consider all of the above arguments in our subsequent calculations.

We start by considering the quadratic corrections that come from the gauge boson Goldstone boson term. We illustrate this only for the Z_μ boson:

$$\begin{aligned}&\langle\int d^4xd^4y(-m_Z\partial^\mu\Phi_3(x)Z_\mu(x))(-im_Z\partial_\nu\Phi_3(y)Z^\nu(y))\rangle = \\ &\delta(0)\frac{3m_Z^2}{16\pi^2}[\Lambda^2 - m_Z^2\ln[\frac{\Lambda^2}{m_Z^2}]]\end{aligned} \tag{1.317}$$

A similar contribution is obtained for the $W_\mu^\pm$ bosons for which we shall write down only the result:

$$\begin{aligned}&\langle\int\int d^4xd^4y2(\frac{-i}{\sqrt{2}}m_W\partial^\mu\Phi'^+(x)W_\mu^-(x))(i\frac{i}{\sqrt{2}}m_W\partial_\nu\Phi'^-(y)W^\nu(y))\rangle = \\ &\delta(0)\frac{6m_W^2}{16\pi^2}[\Lambda^2 - m_W^2\ln[\frac{\Lambda^2}{m_W^2}]].\end{aligned} \tag{1.318}$$

. If we add the contribution in Eqs. (1.317) and (1.318) to the leading one loop quadratic corrections in Eq. (1.315) we obtain:

$$\begin{aligned}-2m_0^2 &= -\frac{3}{2}m_Z^2 - 3m_W^2 + 3m_Z^2 + 6m_W^2 + \frac{m_h^2}{2} \\ -2m_0^2 &= \frac{3}{2}m_Z^2 + 3m_W^2 + \frac{m_h^2}{2},\end{aligned} \tag{1.319}$$

which does not make sense as it is and clearly shows the need for the two loop corrections.

In general a cut-off regularization procedure may be just for one loop corrections but not quite so for higher loops because the translation and gauge invariance are lost. However because we are interested in quadratic corrections we shall proceed further and consider only those Feynman diagrams that are well behaved under a cut-off procedure and neglect those more intricate that require the introduction of the Feynman parameters. Namely the diagrams that we will take into account are those diagrams that may be written as product of two loops or vacuum diagrams that contain two connected bubbles, be they of the tadpole type. Since the diagrams we consider contain product of two traces and those that we ignore contain one trace and mostly Goldstone bosons we estimate that these latter are strongly suppressed and at most may contribute to the final result by one percent.

Below we shall briefly enumerate those two loop contributions that we shall take into account for which we are interested mainly in corrections of order Λ^2.

1) Tadpole diagram involving the top quark and the Z_μ boson:

$$I_1 = \langle \int d^4x d^4y m_Z^2 Z^\mu(x) Z_\mu(x) \frac{h(x)}{v}(-i)\frac{m_t}{v}\bar{t}(y)t(y)h(y)\rangle =$$
$$-36\delta(0)\frac{m_Z^2 m_t^2}{m_h^2 v^2}\frac{1}{256\pi^4}(\Lambda^2 - m_Z^2\ln[\frac{\Lambda^2}{m_Z^2}])(\Lambda^2 - m_t^2\ln[\frac{\Lambda^2}{m_t^2}]). \tag{1.320}$$

2) Tadpole diagram involving the top quark and the $W_\mu^\pm$ boson:

$$I_2 = \langle \int d^4x d^4y 2m_W^2 W^{+\mu}(x) W_\mu^-(x) \frac{h(x)}{v}(-i)\frac{m_t}{v}\bar{t}(y)t(y)h(y)\rangle =$$
$$-72\delta(0)\frac{m_W^2 m_t^2}{m_h^2 v^2}(\Lambda^2 - m_W^2\ln[\frac{\Lambda^2}{m_W^2}])(\Lambda^2 - m_t^2\ln[\frac{\Lambda^2}{m_t^2}]). \tag{1.321}$$

3) Tadpole diagrams that contains four Z_μ bosons:

$$I_3 = \langle \int d^4x d^4y m_Z^2 Z^\mu(x) Z_\mu(x) \frac{h(x)}{v} i m_Z^2 Z^\mu(y) Z_\mu(y)\frac{h(y)}{v}\rangle =$$
$$9\delta(0)\frac{m_Z^4}{m_h^2 v^2}\frac{1}{256\pi^4}(\Lambda^2 - m_Z^2\ln[\frac{\Lambda^2}{m_Z^2}])(\Lambda^2 - m_Z^2\ln[\frac{\Lambda^2}{m_Z^2}]). \tag{1.322}$$

4) Tadpole diagram that contains four $W_\mu^\pm$ bosons:

$$I_4 = \langle \int d^4x d^4y 4m_W^2 W^{\mu+}(x) W_\mu^-(x) \frac{h(x)}{v} i m_W^2 W^{\mu+}(y) W_\mu^-(y)\frac{h(y)}{v}\rangle =$$
$$36\delta(0)\frac{m_W^4}{m_h^2 v^2}\frac{1}{256\pi^4}(\Lambda^2 - m_W^2\ln[\frac{\Lambda^2}{m_W^2}])(\Lambda^2 - m_W^2\ln[\frac{\Lambda^2}{m_W^2}]). \tag{1.323}$$

5) Tadpole diagram that contains two Z_μ bosons and two $W_\mu^\pm$ bosons:

$$I_5 = \langle \int d^4xd^4y4m_W^2 W^{\mu+}(x)W_{\mu-}(x)\frac{h(x)}{v} im_z^2 Z^\mu(y)Z_\mu(y)\frac{h(y)}{v}\rangle =$$
$$36\delta(0)\frac{m_W^2 m_Z^2}{m_h^2 v^2}\frac{1}{256\pi^4}(\Lambda^2 - m_W^2\ln[\frac{\Lambda^2}{m_W^2}])(\Lambda^2 - m_Z^2\ln[\frac{\Lambda^2}{m_Z^2}]). \quad (1.324)$$

6) Tadpole diagrams involving four Higgs bosons and two Z_μ.

$$I_6 = \langle \int d^4xd^4ym_Z^2 Z^\mu(x)Z_\mu(x)\frac{h(x)}{v}\lambda v h^3(y)\rangle =$$
$$9\delta(0)\frac{m_Z^2}{2v^2}\frac{1}{256\pi^4}(\Lambda^2 - m_h^2\ln[\frac{\Lambda^2}{m_h^2}])(\Lambda^2 - m_Z^2\ln[\frac{\Lambda^2}{m_Z^2}]). \quad (1.325)$$

7) Tadpole diagram involving four Higgs bosons and two $W_\mu^\pm$ bosons:

$$I_7 = \langle \int d^4xd^4y2m_W^2 W^{\mu+}(x)W_\mu^-(x)\frac{h(x)}{v}\lambda v h^3(y)\rangle =$$
$$18\delta(0)\frac{m_W^2}{2v^2}\frac{1}{256\pi^4}(\Lambda^2 - m_h^2\ln[\frac{\Lambda^2}{m_h^2}])(\Lambda^2 - m_W^2\ln[\frac{\Lambda^2}{m_W^2}]). \quad (1.326)$$

8) Diagram with two Higgs bosons and two Z_μ:

$$I_8 = \langle \int d^4x\frac{m_Z^2}{2}Z^\mu(x)Z_\mu(x)\frac{h(x)^2}{v^2}\rangle =$$
$$-\frac{3}{2}\delta(0)\frac{m_Z^2}{v^2}\frac{1}{256\pi^4}(\Lambda^2 - m_Z^2\ln[\frac{\Lambda^2}{m_Z^2}])(\Lambda^2 - m_h^2\ln[\frac{\Lambda^2}{m_h^2}]). \quad (1.327)$$

9) Diagram with two Higgs bosons and two $W_\mu^\pm$:

$$I_9 = \langle \int d^4x\frac{m_W^2}{v^2}W^{\mu+}(x)W_\mu^-(x)\frac{h(x)^2}{v^2}\rangle =$$
$$-3\delta(0)\frac{m_W^2}{v^2}\frac{1}{256\pi^4}(\Lambda^2 - m_W^2\ln[\frac{\Lambda^2}{m_W^2}])(\Lambda^2 - m_h^2\ln[\frac{\Lambda^2}{m_h^2}]). \quad (1.328)$$

10) Diagrams with two Z_μ and Goldstone bosons after spontaneous symmetry breaking:

$$I_{10}+I_{11}=\langle\int d^4x\frac{m_Z^2}{2}Z^\mu(x)Z_\mu(x)\frac{\Phi_3^2(x)}{v^2}+\int d^4x\frac{m_Z^2}{2}\frac{b^2}{a^2}Z^\mu(x)Z_\mu(x)\frac{\Phi'^+\Phi'^-}{v^2}\rangle=$$
$$-\frac{3}{2}\delta(0)\frac{m_Z^2}{v^2}\frac{1}{256\pi^4}(\Lambda^2-m_Z^2\ln[\frac{\Lambda^2}{m_Z^2}])\Lambda^2-3\delta(0)\frac{m_Z^2}{v^2}\frac{b^2}{a^2}\frac{1}{256\pi^4}(\Lambda^2-m_Z^2\ln[\frac{\Lambda^2}{m_Z^2}])\Lambda^2 \tag{1.329}$$

11) Diagrams with two $W_\mu^\pm$ and Goldstone bosons after symmetry breaking:

$$I_{12}+I_{13}=\langle\int d^4x\frac{m_W^2}{v^2}W^{\mu+}(x)W_\mu^-(x)\frac{\Phi_3^2}{v^2}+\int d^4x\frac{m_W^2}{v^2}W^{\mu+}(x)W_\mu^-(x)\frac{\Phi'^+\Phi'^-}{v^2}\rangle=$$
$$-3\delta(0)\frac{m_W^2}{v^2}\frac{1}{256\pi^4}(\Lambda^2-m_W^2\ln[\frac{\Lambda^2}{m_W^2}])\Lambda^2-6\delta(0)\frac{m_W^2}{v^2}\frac{1}{256\pi^4}(\Lambda^2-m_W^2\ln[\frac{\Lambda^2}{m_W^2}])\Lambda^2 \tag{1.330}$$

12) Diagrams with two Z_μ and the four Higgses before spontaneous symmetry breaking:

$$I_{14}+I_{15}=\langle\int d^4x\frac{m_Z^2}{v^2}Z^\mu(x)Z_\mu(x)\frac{\Phi_0^2(x)}{v^2}+\int d^4x\frac{m_Z^2}{v^2}\frac{b^2}{a^2}Z^\mu(x)Z_\mu(x)\frac{\Phi^+\Phi^-}{v^2}\rangle=$$
$$-3\delta(0)\frac{m_Z^2}{v^2}\frac{1}{256\pi^4}(\Lambda^2+m_0^2\ln[\frac{\Lambda^2}{m_0^2}])\Lambda^2-3\delta(0)\frac{m_Z^2}{v^2}\frac{b^2}{a^2}\frac{1}{256\pi^4}(\Lambda^2+m_0^2\ln[\frac{\Lambda^2}{m_0^2}])\Lambda^2. \tag{1.331}$$

13) Diagrams with two $W_\mu^\pm$ and the four Higgses before spontaneous symmetry breaking:

$$I_{16}=\langle\int d^4x2\frac{m_W^2}{v^2}W^{\mu+}(x)W_\mu^-(x)\frac{\Phi_0^2}{v^2}+\int d^4x2\frac{m_W^2}{v^2}W^{\mu+}(x)W_\mu^-(x)\frac{\Phi'^+\Phi'^-}{v^2}\rangle=$$
$$-3\delta(0)\frac{m_W^2}{v^2}\frac{1}{256\pi^4}4(\Lambda^2+m_0^2\ln[\frac{\Lambda^2}{m_0^2}])\Lambda^2. \tag{1.332}$$

In what follows we shall ignore corrections to the gauge boson and fermion masses and replace the bare masses of these with the physical ones. In the terms involving the Higgs propagator we shall just replace the bare mass by the physical one as the pole of the propagator is at the physical mass and neglect higher order corrections. In the rest of the formula we shall however substitute the bare Higgs mass by the one loop corrected one according to:

$$m_h^2=2m_0^2+\Sigma(m_h^2), \tag{1.333}$$

where,

$$\Sigma(m_h^2)=\frac{3}{8\pi^2v^2}[4m_t^2-m_h^2-2m_W^2-m_Z^2]\Lambda^2-$$
$$\frac{3}{16\pi^2}\frac{m_h^2}{v^2}[m_h^2\ln[\frac{\Lambda^2}{m_h^2}]-2m_W^2\ln[\frac{\Lambda^2}{m_W^2}]-m_Z^2\ln[\frac{\Lambda^2}{m_Z^2}]+2m_t^2\ln[\frac{\Lambda^2}{m_t^2}]]. \tag{1.334}$$

For simplicity we shall denote:

$$\Sigma(m_h^2) = x\Lambda^2 + y\ln[\Lambda^2] + z, \tag{1.335}$$

where x is the coefficient of the quadratic divergent term, y that of the logarithmic one and z is the constant contribution. Note that whenever we use the logarithm of a quantity with mass dimension m^2 expressed in GeV^2 the division of the argument of the logarithm by 1 GeV^2 is implicitly assumed.

Now we add all two loop contributions in the Eqs. (1.320)-(1.332) to the one loop contribution in Eq. (1.319) and apply Eq. (1.333) to obtain as a result of equating the gauge Higgs kinetic term before and after spontaneous symmetry breaking:

$$\begin{aligned}&\frac{3}{2}(m_h^2 - z) - x(m_h^2 - z)\ln[2] + x(\frac{m_h^2}{2} - z)\ln[m_h^2] = \\ &= -3m_W^2 - \frac{3}{2}m_Z^2 - \sum_{j=1}^{16}(16\pi^2)(I_j)_{\Lambda^2}\frac{1}{\delta(0)},\end{aligned} \tag{1.336}$$

where $(I_j)_{\Lambda^2}$ denotes the coefficient of the quadratic divergence in the integral I_j.

In this subsection we made the hypothesis that the quantum correlators associated to the space time integral of gauge invariant kinetic terms of the standard model Higgs boson doublet for the symmetric Lagrangian (partition function) and spontaneously broken Lagrangian (partition function) are equal. This assumption is based on the fact that if one uses the same gauge fixing functions for the Lagrangian before and after spontaneous symmetry breaking the only change that occurs in the Lagrangian and partition function at the symmetry breaking point is a change of variable by a constant shift of the neutral component of the Higgs doublet.

We computed the corresponding quantities and their quadratic contribution at one loop with corrections from the most relevant diagrams at two loop.

In Fig.1 we plotted C_1 and C_2 where:

$$\begin{aligned}C_1 &= \frac{3}{2}(m_h^2 - z) - x(m_h^2 - z)\ln[2] + x(\frac{m_h^2}{2} - z)\ln[m_h^2] \\ C_2 &= -3m_W^2 - \frac{3}{2}m_Z^2 - \sum_{j=1}^{16}(16\pi^2)(I_j)_{\Lambda^2}\frac{1}{\delta(0)},\end{aligned} \tag{1.337}$$

to find that the two curves intersect for a mass of the Higgs boson $m_h \approx 125.07$ GeV. This result is strikingly close to the experimental mass (of the Higgs boson $m_{hexp} = 125.09 \pm 0.24$

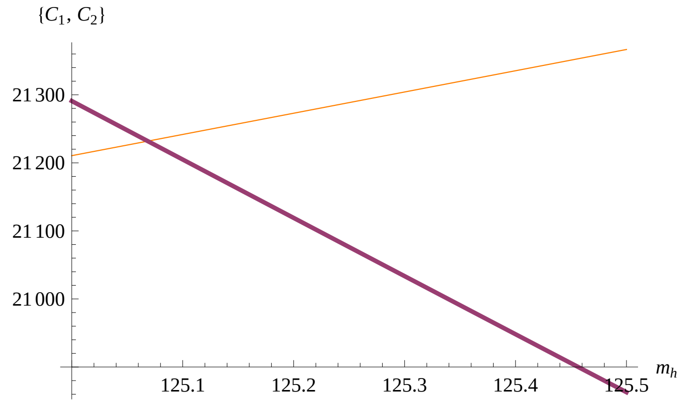

FIG. 2: Plot of the quantities C_1 (GeV^2) and C_2 (GeV^2) as a function of the Higgs boson mass m_h (GeV).

GeV. We estimate the other two loop diagrams contribution or corrections to the masses of the other particles involved besides the Higgs for at most 1 percent of the value obtained in this work. Our finding suggest that our initial assumption may be well justified.

If a beyond the standard model theory is employed instead then our hypothesis should apply to each gauge invariant kinetic operator for the scalars involved and the results should be greatly altered due to different possible combinations of vacuum expectation values and masses. Consequently it is very possible that the electroweak breaking sector of the standard model might contain at least in first order as an effective theory a single Higgs doublet with the well known vacuum expectation value and Goldstone bosons.

II. PAPERS REPORTED FOR THE PROJECT IN THE PERIOD SEPTEMBER 2013-DECEMBER 2016

1. "About the role of scalars in a gauge theory", R. Jora, S. Nasri, arXiv: 1310.6122, Rom. J. Phys. 60, 9-10 (2015).
2. "A hierarchy of the quark masses in a top condensate model with multiple Higgses", A. H. Fariborz, R. Jora, S. Nasri, J. Schechter, arXiv:1310.1721, Modern Physics Letters A 29, 6, 1450030 (2014).
3. "A nonperturbative method for the scalar field theory", R. Jora, arXiv:1403.2227

(2014), Physics Letters B 741, 223-231 (2014).

4. "A hint of a strong supersymmetric standard model", R. Jora and J. Schechter, arXiv:1403.3778 (2014), Int. J. Mod. Phys. A 29, 31, 1460192 (2014).

5. "A semi perturbative method for QED", R. Jora and J. Schechter, arXiv:1407.8172 (2014), Rom. J. Phys. 60, 7-8 (2015).

6. "Tree level relations in a composite electroweak theory", A. H. Fariborz, R. Jora and J. Schechter, arXiv:1407.6546 (2014), Mod. Phys. Lett. A 29, 31, 1450166.

7. "Probing scalar mesons in semi-leptonic decays of D_s^+, D^+ and D^0", A. H. Fariborz, R. Jora, J. Schechter, M.N. Shahid, arXiv:1407.7176 (2014), Int. J. Mod. Phys. A 30 02, 1550012 (2015).

8. "A semiclassical approach for the Higgs boson", A.H. Fariborz, R. Jora, J. Schechter, arXiv:1409.7886 (2014), Int. J. Mod. Phys. A 29, 32, 1450196.

9. "A nonperturbative method for the Yang Mills Lagrangian", R. Jora, arXiv:1411.0211 (2014), Int. J. Mod. Phys. A 30 13, 1550070 (2015).

10. "Abelian Higgs model and its phase transitions revisited", R. Jora, arXiv: 1502.04293 (2015), in press, Romanian Reports in Physics.

11. "How important is $i\epsilon$ in QFT", A. H. Fariborz, R. Jora and J. Schechter, arXiv:1503.00101 (2015), Int. J. Mod. Phys. A 30 14, 1550071 (2015).

12. "Φ^4 Theory is trivial", R. Jora, arXiv: 1503.07298 (2015), Rom. J. Phys. 61, no. 3-4, 314 (2015).

13. "Phase transitions for SU(N) gauge theories with arbitrary number of flavors", R. Jora, arXiv: 1505.07274, Int. J. Mod. Phys. A 30, no. 34, 1550203 (2015).

14. "A solution to the naturalness problem", A. H. Fariborz and R. Jora, arXiv: 1506.00079 (2015), accepted by Rom. J. Phys.

15. "A nonperturbative method for QCD", R. Jora, arXiv:1506.06473 (2015), Rom. J. Phys. 61, no. 5-6, 755 (2016).

16. "A purely quark Lagrangian for QCD", A. H. Fariborz and R. Jora, arXiv:1509.05993 (2015), in evaluation.

17. "Note on the standard model as an effective theory", A. H. Fariborz and R. Jora, arXiv: 1509.05993 (2015), in evaluation.

18. "About the particle structure and content of the standard model", R. Jora, arXiv: 1510.08223 (2015), in evaluation.

19. "Vacuum energy in some particular quantum field theories", R. Jora, arXiv: 1602.06146, invited paper by the Proc. Rom. Academy.

20. "About the mass corrections in an abelian Higgs model", R. Jora, arXiv: 1603.04192, Rom. Rep. Phys 68, no. 3, 935-942 (2016).

21. "Spontaneous symmetry breaking and Goldstone theorem for composite states revisited", A. H. Fariborz and R. Jora, atXiv:1607.01199 (2016), in evaluation.

22. "Higgs boson mass from gauge invariant operators", R. Jora, arXIv:1607.01949 (2016), in evaluation.

23. "Mass Ansatze for the standard model fermions from a composite perspective", A. H. Fariborz, R. Jora and S. Nasri, arXiv:1608.07414 (2016), accepted by Int. J. Mod. Phys. A: ATP.

24. "Examining a possible cascade effect in chiral symmetry breaking", A.H. Fariborz and R. Jora, arXiv: 1608.07907 (2016), accepted by Mod. Phys. Lett. A.

25. "About the electroweak vacuum and its connection with the quark condensates", R. Jora, arXiv: 1611.03822 (2016), in evaluation.

In conclusion all the objectives and subsequent activities for the years 2013, 2014,2015, 2016 have been fully achieved.

Project Director

Catalina Renata Jora